Bulk of random matrices generated by Markov chains with community structure

Presented at the fourth ZiF Bielefeld summer school (2022)

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Motivation

Many real-world systems can be modelled as a Markov chain:

- Sequence of words in text
- DNA
- Taxi pickup locations.

The state space can be very large. This causes problems such as algorithms slowing down.

Solution: cluster together similar states.

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Provable clustering

For so-called *block Markov chains* the consistency of spectral community detection algorithms has been proved in [Sanders et. al., 2020] and [Zhang et. al., 2020]. An impossibility theorem when insufficient data is available was also proved in [Sanders et. al., 2020].

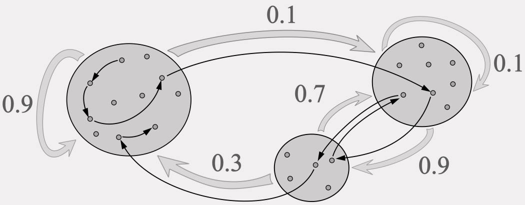
Key references:

- Anru Zhang and Mengdi Wang. Spectral state compression of Markov processes. IEEE Transactions on Information Theory, 66(5):3202–3231, 2020.
- Jaron Sanders, Alexandre Proutière, and Se–Young Yun. Clustering in block Markov chains. The Annals of Statistics, 48(6):3488–3512, 2020.

Block Markov Chain

Fix the following data:

- Number of clusters $K \geq 1$.
- Cluster transition matrix $p \in \mathbb{R}^{K \times K}$.



Then, the *block Markov chain* associated to a partition $\mathcal{V}_1 \cup \ldots \cup \mathcal{V}_K = \{1, \ldots, n\}$ is the Markov chain with transition matrix P given by

$$P_{i,j} = \frac{p_{x,y}}{\#\mathcal{V}_y} \qquad \forall i \in \mathcal{V}_x,\ \forall j \in \mathcal{V}_y.$$

We consider the regime where $n \to \infty$ but K and p are fixed.

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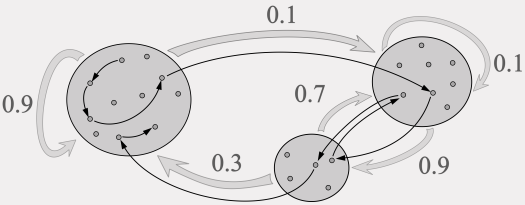

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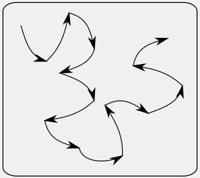
Spectral algorithm

Spectral community detection algorithm:

1. Observe a sample path $X_0, \ldots, X_\ell$.
2. Construct $\widehat{P}_X$, the estimator for the transition matrix P.
3. Principal component analysis of $\widehat{P}_X$ yields embedding of state space into $\mathbb{R}^K$.
4. k-means + improvement algorithm.



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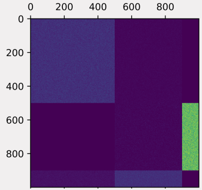

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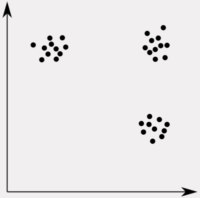

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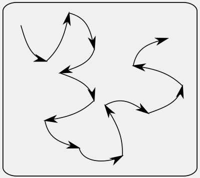

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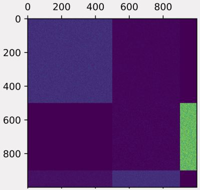

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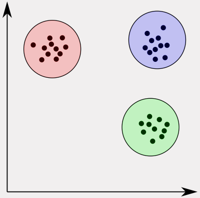

Motivating question

Note that the spectral algorithm relied on singular vectors of $\widehat{P}_X$. This motivates studying the spectral properties of $\widehat{P}_X$ in general. For instance, what is the limiting empirical singular value distribution?

Key difficulty: $\widehat{P}_X$ was constructed based on a sample path of a Markov chain. This is **dependent** data.

Key contribution: A more natural way to deal with the dependence than was used in previous works about block Markov chains. Namely, a coupling argument.

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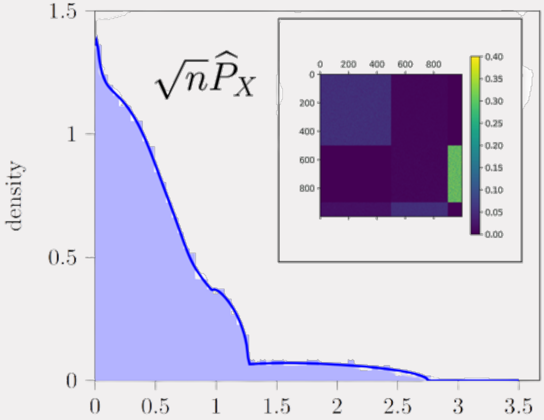
Result



$\sqrt{n}\widehat{P}_X$
density
1.5
1
0.5
0
0
0.5
1
1.5
2
2.5
3
3.5
0
200
400
600
800
0.40
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00

Theorem

Assume that as the size of the state space n tends to infinity it holds that $\#\mathcal{V}_i = \alpha_i n + o(n)$ and that the length of the sample path satisfies $\ell = \lambda n^2 + o(n^2)$. Assume moreover that p defines an ergodic chain with equilibrium distribution π.

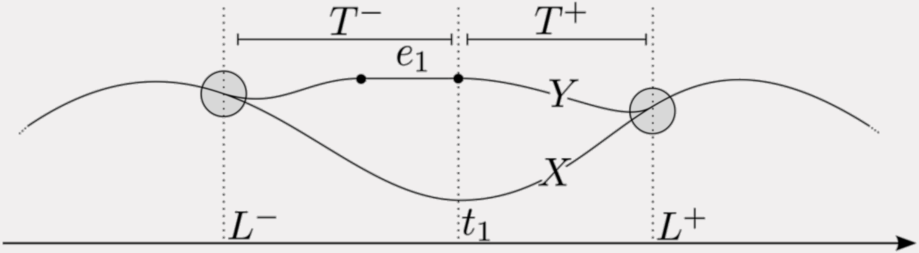
Then, the empirical singular value distribution $\nu_{\sqrt{n}\widehat{P}_X}$ converges weakly in probability to a probability measure ν on $\mathbb{R}_{\geq 0}$ whose symmetrization $\mathrm{sym}(\nu)$ has Stieltjes transform $s(z) = \sum_{i=1}^{K} \alpha_i (a_i(z) + a_{K+i}(z))/2$ with for $i = 1, \dots, K$

$$a_i(z)^{-1} = z - \sum_{j=1}^{K} \lambda^{-1} \pi(i)^{-1} \alpha_i p_{i,j} a_{K+j}(z)$$

$$a_{i+K}(z)^{-1} = z - \sum_{j=1}^{K} \lambda^{-1} \pi(j)^{-1} \alpha_i^{-1} \alpha_j^2 p_{j,i} a_j(z).$$

Key proof technique: Coupling

One can pretend as if the observation at time t is independent of all other observations up to some error term. More precisely, given $t_1 \in \{0, \ldots, \ell\}$ one constructs a path Y with Y independent of X_{t_1} such that $Y_t = X_t$ for most $t \neq t_1$.



Thank you!

The key references for this talk are:

- The current result: *Jaron Sanders and Alexander Van Werde. "Singular value distribution of dense random matrices with block Markovian dependence." arXiv preprint arXiv:2204.13534 (2022).*
- Spectral algorithm and information-theoretic lower bound: *Jaron Sanders, Alexandre Proutière, and Se–Young Yun. Clustering in block Markov chains. The Annals of Statistics, 48(6):3488–3512, 2020.*
- Different perspective on community detection in Markov chains and an algorithm: *Anru Zhang and Mengdi Wang. Spectral state compression of Markov processes. IEEE Transactions on Information Theory, 66(5):3202–3231, 2020.*

A different approach to coupling in dependent random matrices, namely a blocking procedure, can be found in the works of Merlevède and Banna in the period 2015-2016.