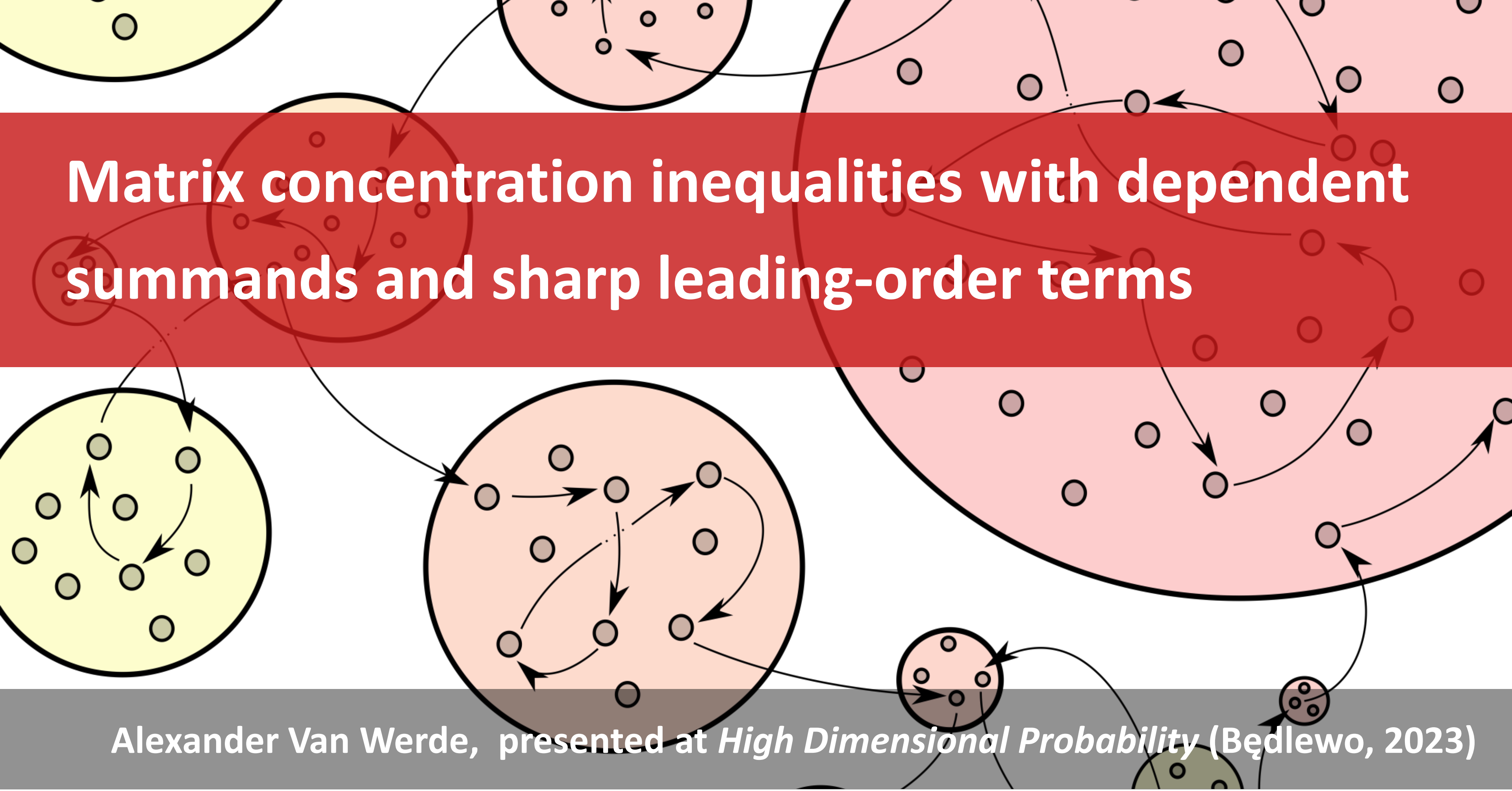

# Matrix concentration inequalities with dependent summands and sharp leading-order terms

Alexander Van Werde, presented at *High Dimensional Probability* (Będlewo, 2023)

Joint work with Jaron Sanders

TU/e EINDHOVEN UNIVERSITY OF TECHNOLOGY

# Part I

# Context & related work

# Matrix Bernstein inequality

**Theorem. (Tropp, Oliveira)**

Let $\boldsymbol{X}_1, \dots, \boldsymbol{X}_n$ be independent self-adjoint $d \times d$ matrices.
Assume $\mathbb{E}\boldsymbol{X}_i = 0$ and $\|\boldsymbol{X}_i\| \le R$. Denote $\boldsymbol{S} := \sum_{i=1}^{n} \boldsymbol{X}_i$.

Then, for some absolute $c > 0$, the operator norm satisfies

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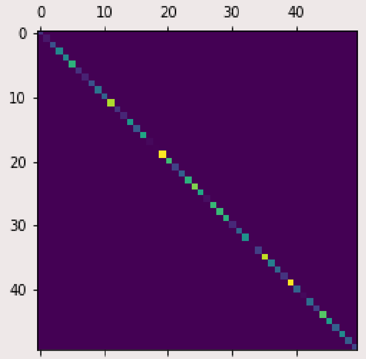

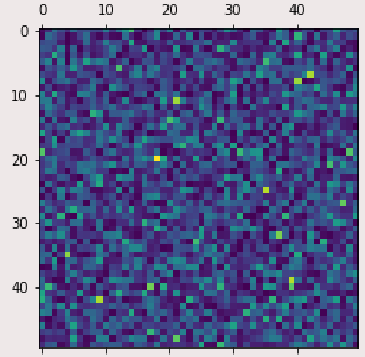

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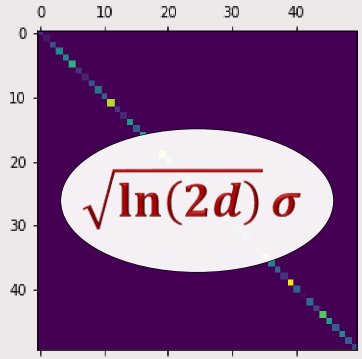

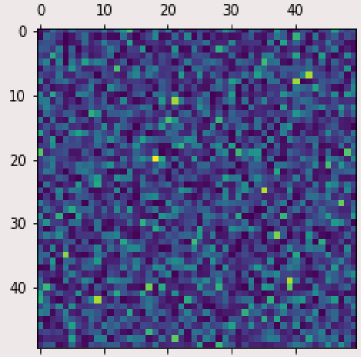

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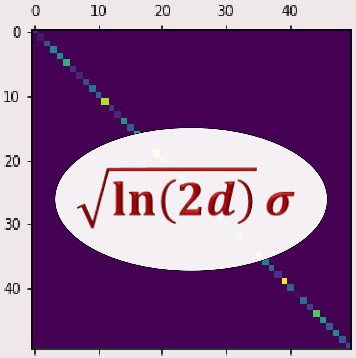

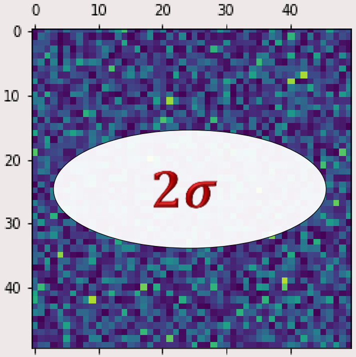

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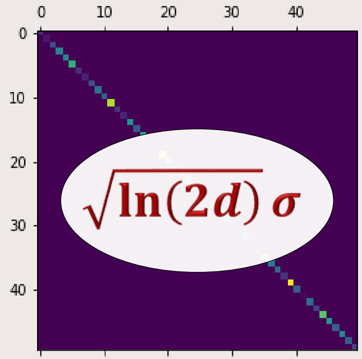

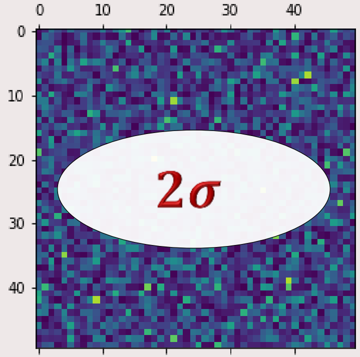

# Some related literature

## Matrix concentration with dependent summands

- **R.I. Oliveira (2009):**
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- **J. Neeman, B. Shi, and R. Ward (2023):**
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# Part II

## Sharp matrix concentration

### for Markovian model & matrix series model

# Markovian model and parameters

**Markovian model.**

Consider a Markov chain $Z_1, \ldots, Z_n$.

Generate self-adjoint $d \times d$ matrices

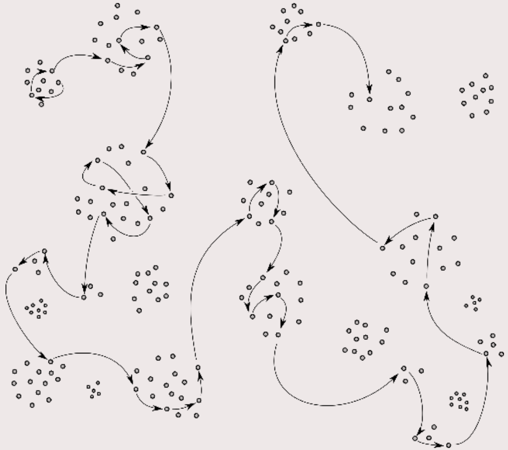
$$\boldsymbol{X}_i := f_i(Z_i) \quad \text{and} \quad \boldsymbol{S} := \sum_{i=1}^{n} \boldsymbol{X}_i$$

Further, for simplicity, also assume

- Finite state space
- Transition matrix $P$
  & equilibrium distribution $\pi$
- $\mathbb{E}\boldsymbol{X}_i = 0$

**Dependence parameter**

$$\Psi := \min\{\, t \geq 1 : \frac{|P_{x,y}^{t} - \pi_y|}{\pi_y} \leq \frac{1}{4} \ \forall x, y\}$$



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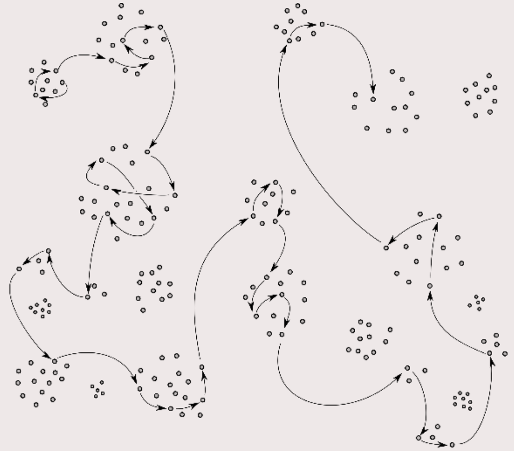

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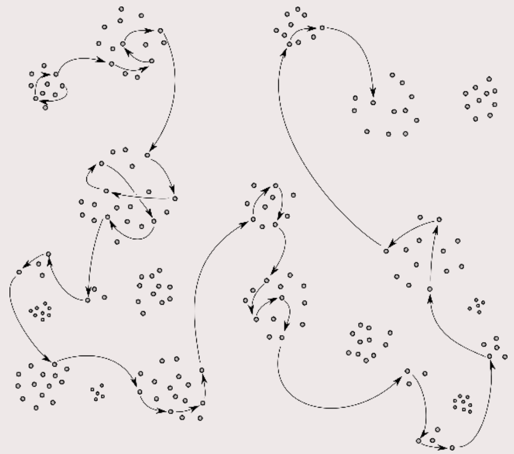

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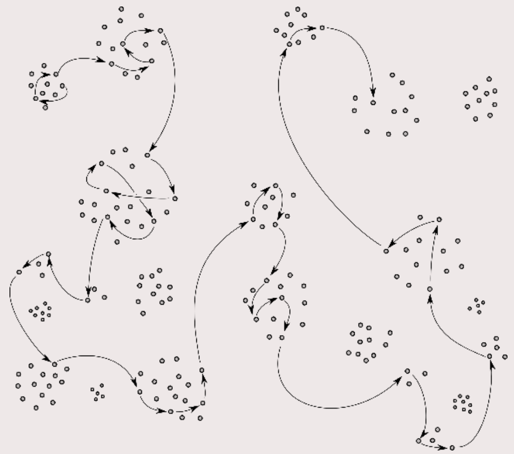

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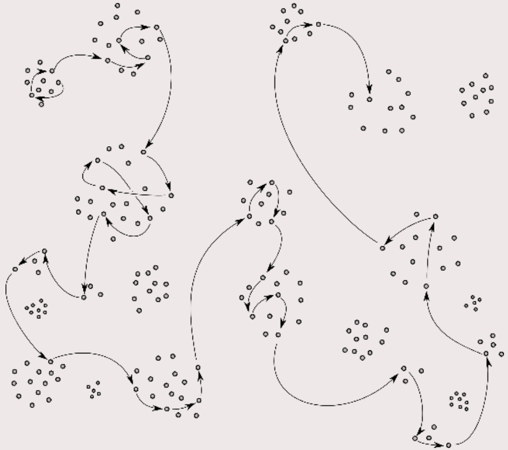

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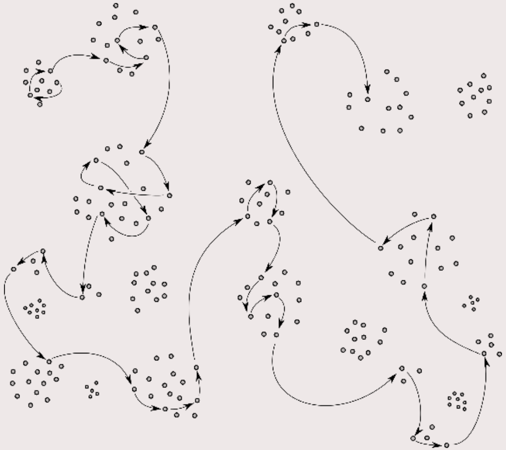

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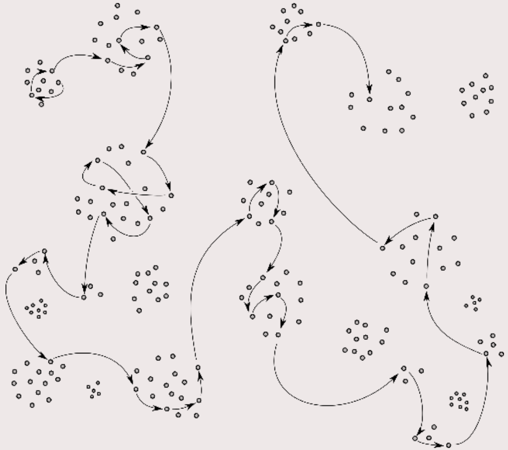

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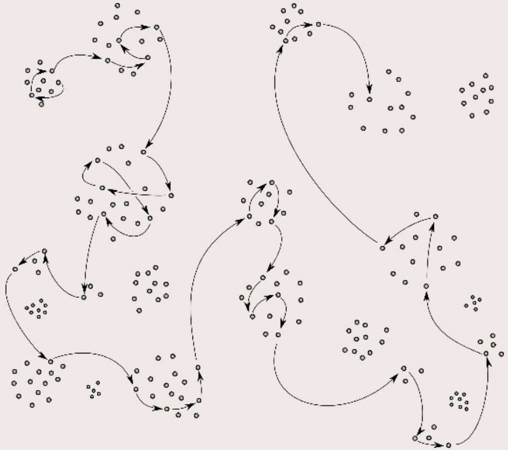

# Sharp matrix concentration in Markovian model

**Main result.**

There exists $c > 0$ such that for $0 < \delta \leq 1$

$$\mathbb{E}\|\boldsymbol{S}\| \leq (1+\delta)\|\boldsymbol{S}_{\mathrm{free}}\| + c\mathcal{E}_{d,\delta}$$

where

$$\begin{aligned}\mathcal{E}_{d,\delta} := {} & v^{1/2}\sigma^{1/2}\log_{1+\delta}(2d)^{3/4} \\ & + R^{1/3}\Psi^{2/3}\varsigma^{2/3}\log_{1+\delta}(2d)^{2/3} \\ & + R\,\Psi\log_{1+\delta}(2d)\end{aligned}$$

**Dependence parameter**

$$\Psi := \min\{\, t \geq 1 \colon \frac{|P^t_{x,y} - \pi_y|}{\pi_y} \leq \frac{1}{4} \ \forall x, y\}$$

**Free-probabilistic quantity**

$$\|\boldsymbol{S}_{\mathrm{free}}\| = \inf_{\boldsymbol{W} \succ \boldsymbol{0}} \lambda_{\max}\left(\boldsymbol{W}^{-1} + \mathbb{E}[\boldsymbol{S}\boldsymbol{W}\boldsymbol{S}]\right).$$

**Matrix parameters**

$$\varsigma^2 := \left\|\mathbb{E}\left[\textstyle\sum_{i=1}^n \boldsymbol{X}_i^2\right]\right\|, \quad \sigma^2 := \|\mathbb{E}[\boldsymbol{S}^2]\|$$

$$v^2 := \|\mathrm{Cov}(\boldsymbol{S})\|, \quad R \geq \|\boldsymbol{X}_i\|.$$

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**Main result.**

There exists $c > 0$ such that for $0 < \delta \leq 1$

$$\mathbb{E}\|\boldsymbol{S}\| \leq (1+\delta)\|\boldsymbol{S}_{\text{free}}\| + c\mathcal{E}_{d,\delta}$$

where

$$\mathcal{E}_{d,\delta} := v^{1/2}\sigma^{1/2}\log_{1+\delta}(2d)^{3/4}$$
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$$+\ R\,\Psi\log_{1+\delta}(2d)$$

**Dependence parameter**

$$\Psi := \min\{\, t \geq 1 : \frac{|P^t_{x,y} - \pi_y|}{\pi_y} \leq \frac{1}{4}\ \forall x, y\}$$

**Free-probabilistic quantity**

$$\|\boldsymbol{S}_{\text{free}}\| = \inf_{\boldsymbol{W} \succ \boldsymbol{0}} \lambda_{\max}\left(\boldsymbol{W}^{-1} + \mathbb{E}[\boldsymbol{S}\boldsymbol{W}\boldsymbol{S}]\right).$$

**Matrix parameters**

$$\varsigma^2 := \left\|\mathbb{E}\left[\textstyle\sum_{i=1}^n \boldsymbol{X}_i^2\right]\right\|, \quad \sigma^2 := \|\mathbb{E}[\boldsymbol{S}^2]\|$$

$$v^2 := \|\mathrm{Cov}(\boldsymbol{S})\|, \quad R \geq \|\boldsymbol{X}_i\|.$$

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# Part III

# Proof sketch

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**Goal.** $\mathbb{E}[\mathrm{tr}\, \mathbf{S}^{2p}] \approx \mathbb{E}[\mathrm{tr}\, \mathbf{G}^{2p}]$ where $\boldsymbol{G}$ is a *Gaussian model* for **S**

**Interpolate:** For $t \in [0,1]$ set

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**Direct modification using classical cumulants gives suboptimal results…**

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# Part IV

# Illustrative applications

# Literature: Block Markov chains

Spectral clustering in block Markov chains

- **J. Sanders, A. Proutière, S.-Y. Yun (2020):**
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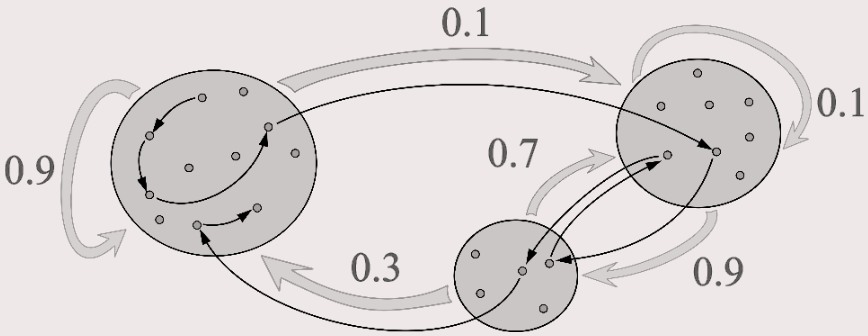
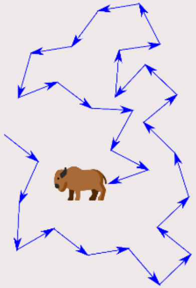
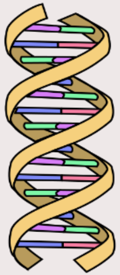
  Real-world data



Animal movement



DNA



Text

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Stock market

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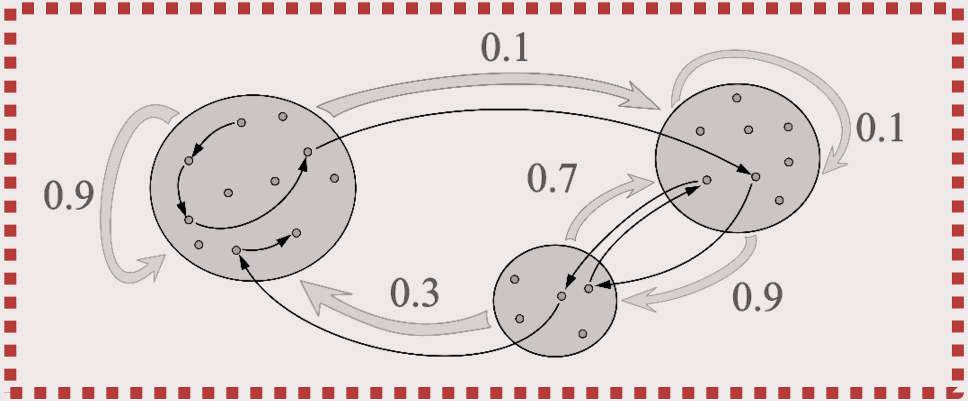

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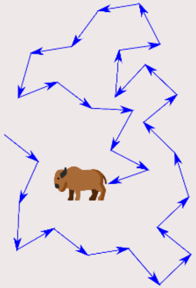

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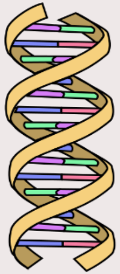

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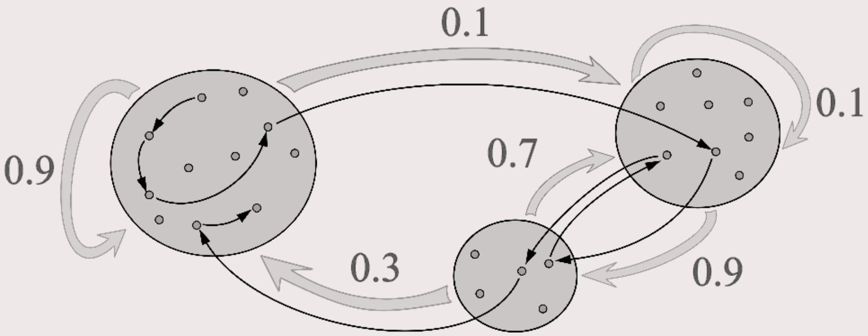
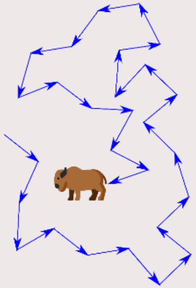
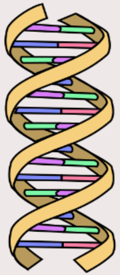


Animal movement



DNA



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Stock market

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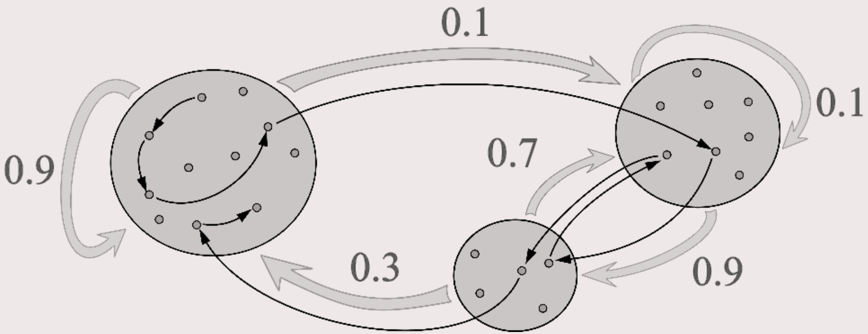

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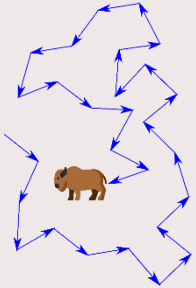

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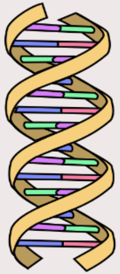

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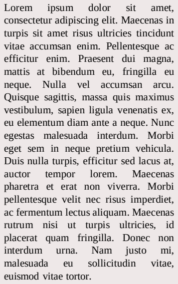

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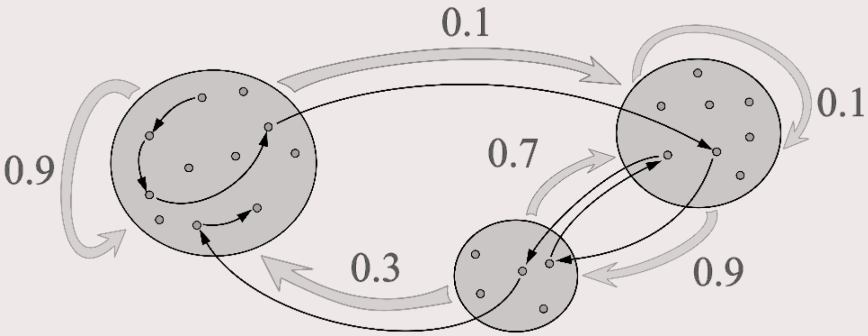

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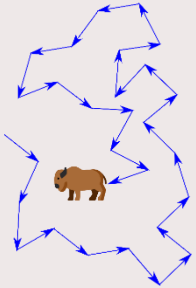

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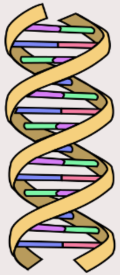

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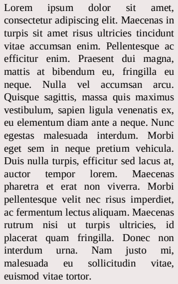

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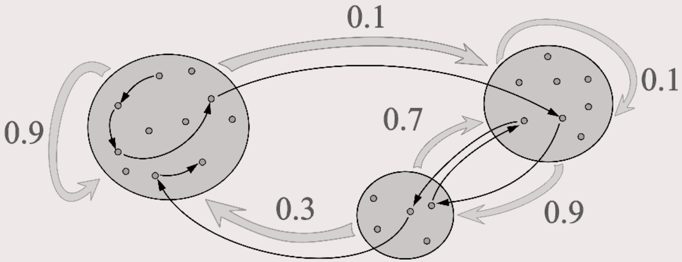

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A *Block Markov chain* has transition matrix

$$P_{i,j} = \frac{p_{x,y}}{\#\mathcal{V}_y} \qquad \forall i \in \mathcal{V}_x, j \in \mathcal{V}_y$$

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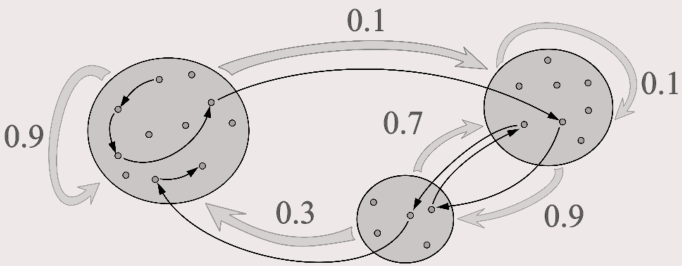

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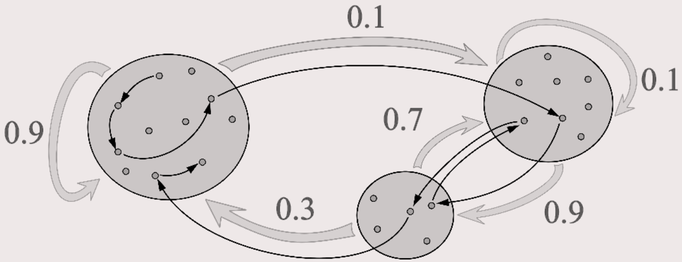

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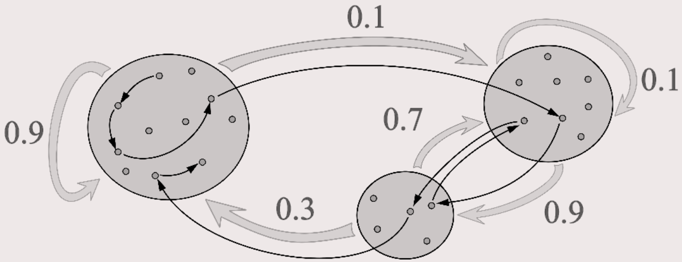

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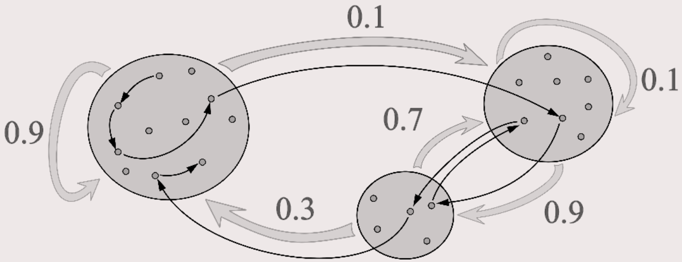

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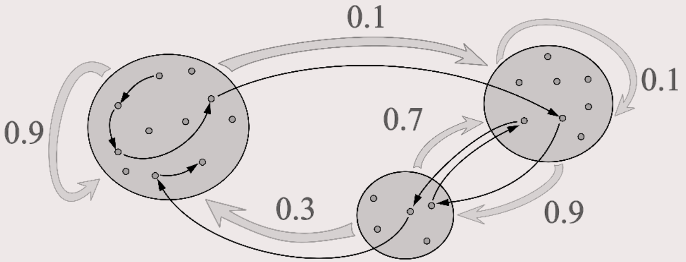

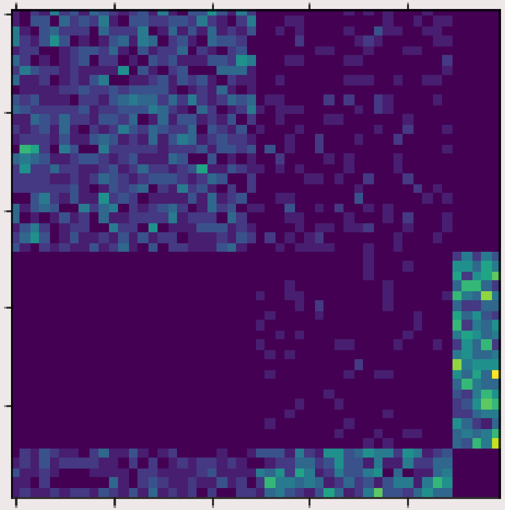

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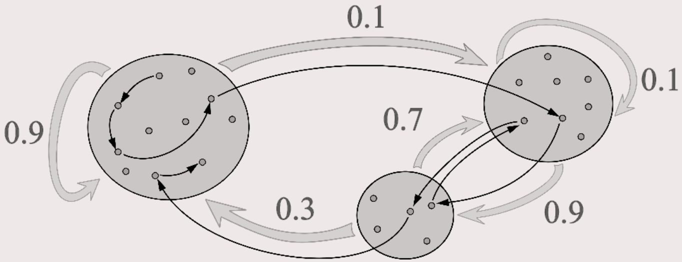

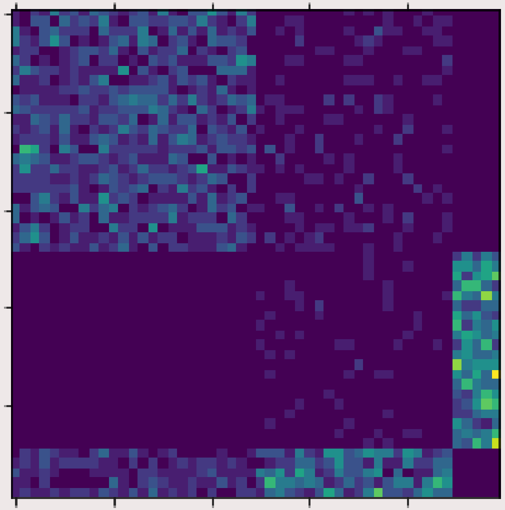

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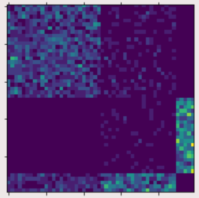

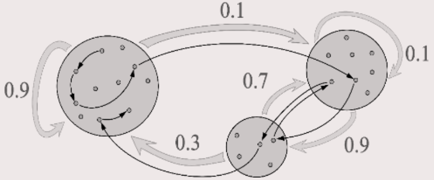

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**Corollary.**

Suppose $n \gg d\ln(d)^4$ as $d \to \infty$, then

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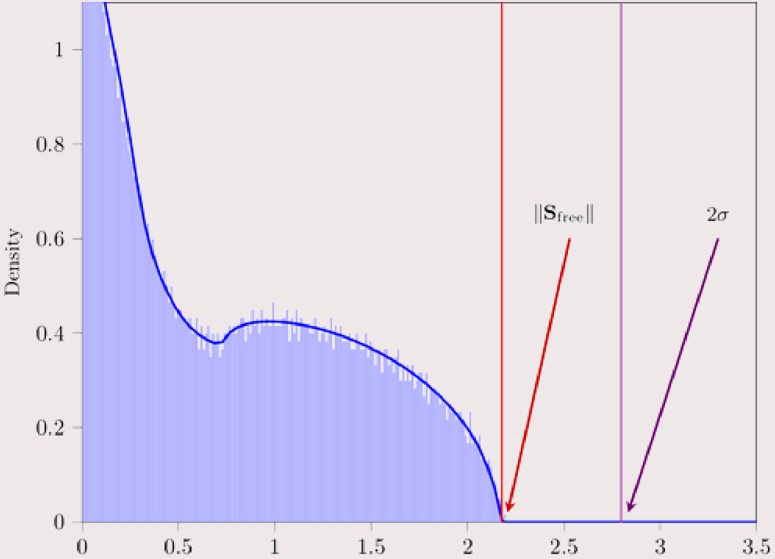

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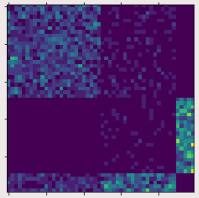

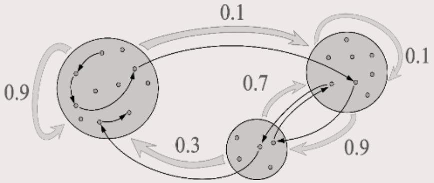

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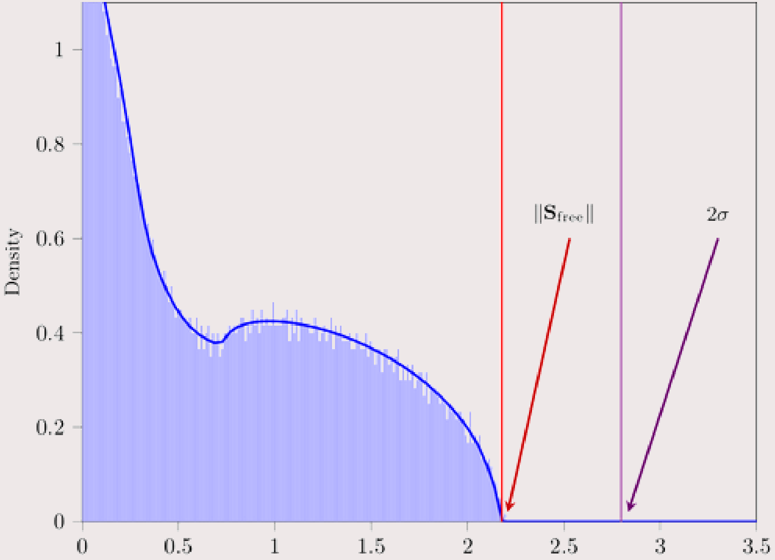

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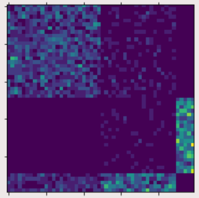

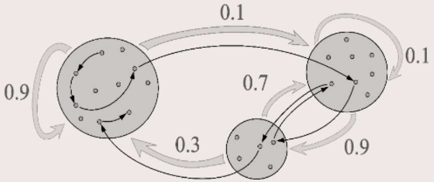

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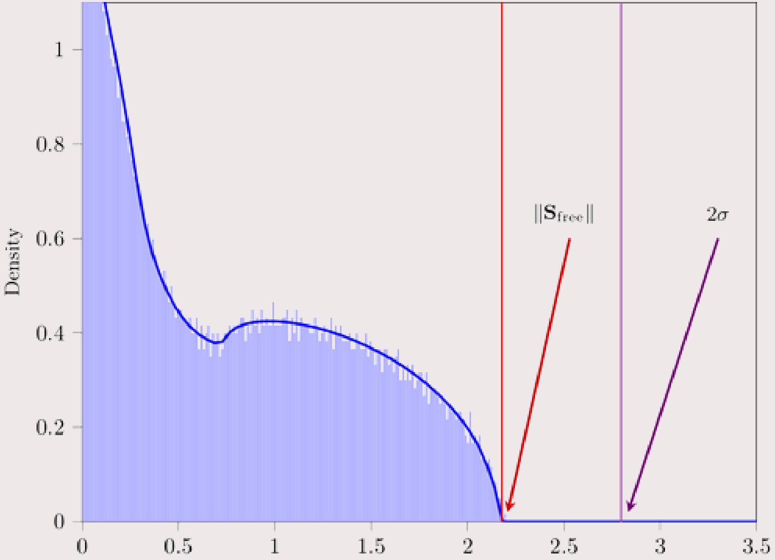

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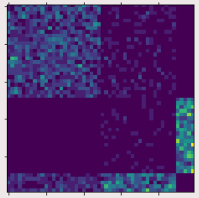

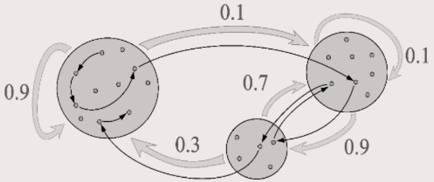

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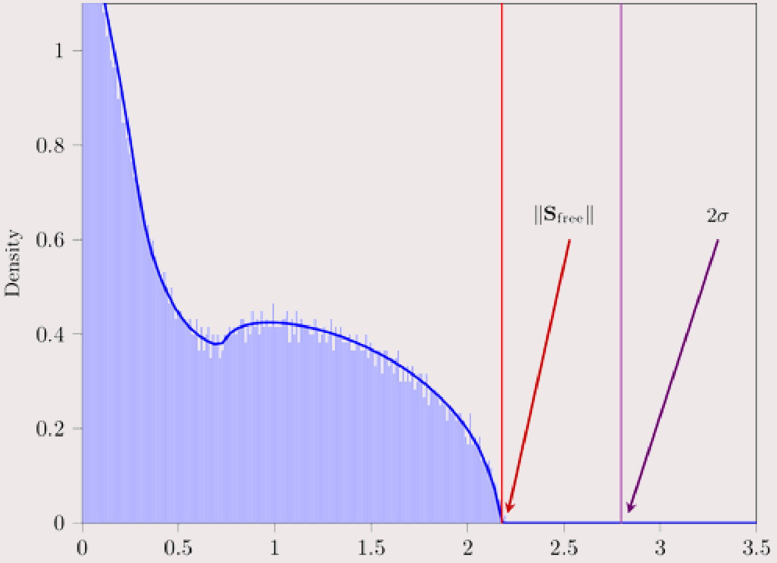


# Applications of matrix series model

**Heavy-tailed independent entries**

Let $\boldsymbol{S}$ be a Wigner matrix with independent sub-Weibull entries.

Then, we show that

$$\|\boldsymbol{S}\| \leq 2\sigma + \varepsilon \text{ with high probability.}$$

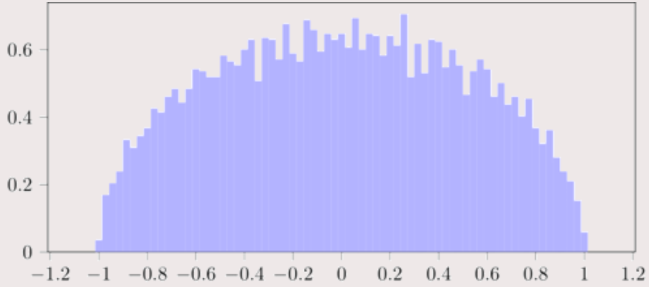
We provide explicit tail bounds.

This is a quantitative non-asymptotic variant on the Bai-Yin theorem.

**Spectra of graphs with dependent edges**

Let $\boldsymbol{A}$ be the adjacency matrix of an Erdős–Rényi random graph with $m$ edges and $d$ nodes.

We show convergence of empirical eigenvalue distributions when $m = \omega(d)$.



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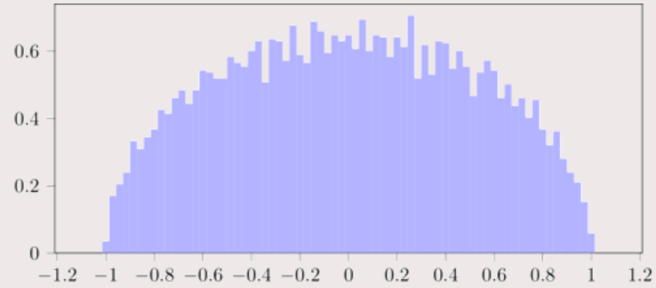

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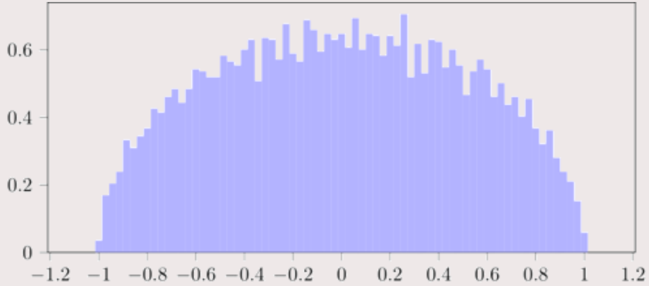

# Summary

We establish ***sharp*** matrix concentration inequalities for sum of ***dependent*** random matrices.

Boolean cumulants play a crucial role in the proof. Surprisingly, classical cumulants are insufficient for sharp estimates.

Sharp concentration in applications, such as block Markov chains, would be inaccessible by previous nonasymptotic dependent results.

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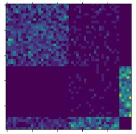
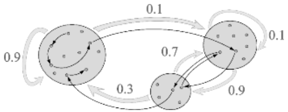
**Appears on arXiv soon!**

# Thank you!





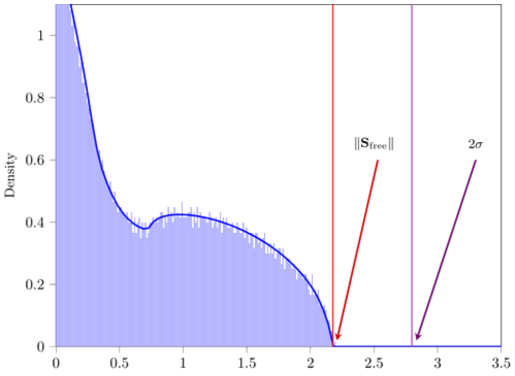
**Main result for Markovian model.**

There exists $c > 0$ such that for any $0 < \delta \leq 1$

$$\mathbb{E}\|\boldsymbol{S}\| \leq (1+\delta)\|\boldsymbol{S}_{\text{free}}\| + c\mathcal{E}_{d,\delta}$$

where

$$\mathcal{E}_{d,\delta} \coloneqq v^{1/2}\sigma^{1/2}\log_{1+\delta}(2d)^{3/4} + R^{1/3}\Psi^{2/3}\varsigma^{2/3}\log_{1+\delta}(2d)^{2/3} + R\Psi\log_{1+\delta}(2d)$$



**Dependence parameter**

$$\Psi \coloneqq \min\{\, t \geq 1 \colon \frac{|P^t_{x,y} - \pi_y|}{\pi_y} \leq \frac{1}{4} \}$$

**Matrix parameters**

$$\varsigma^2 \coloneqq \left\|\mathbb{E}\left[\textstyle\sum_{i=1}^n \boldsymbol{X}_i^2\right]\right\|, \quad \sigma^2 \coloneqq \|\mathbb{E}[\boldsymbol{S}^2]\|$$

$$v^2 \coloneqq \|\mathrm{Cov}(\boldsymbol{S})\|, \quad R \geq \|\boldsymbol{X}_i\|$$

**Free-probabilistic quantity**

$$\|\boldsymbol{S}_{\text{free}}\| = \inf_{\boldsymbol{W}\succ\boldsymbol{0}} \lambda_{\max}\left(\boldsymbol{W}^{-1} + \mathbb{E}[\boldsymbol{S}\boldsymbol{W}\boldsymbol{S}]\right).$$

a.van.werde@tue.nl