

# Matrix concentration inequalities with dependent summands and sharp leading-order terms

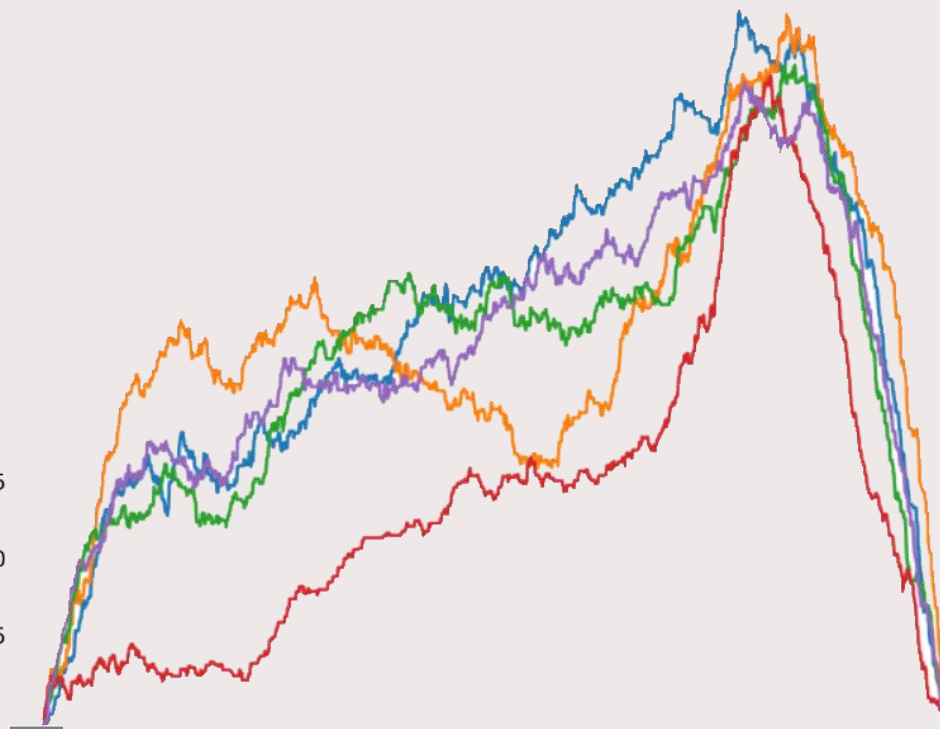
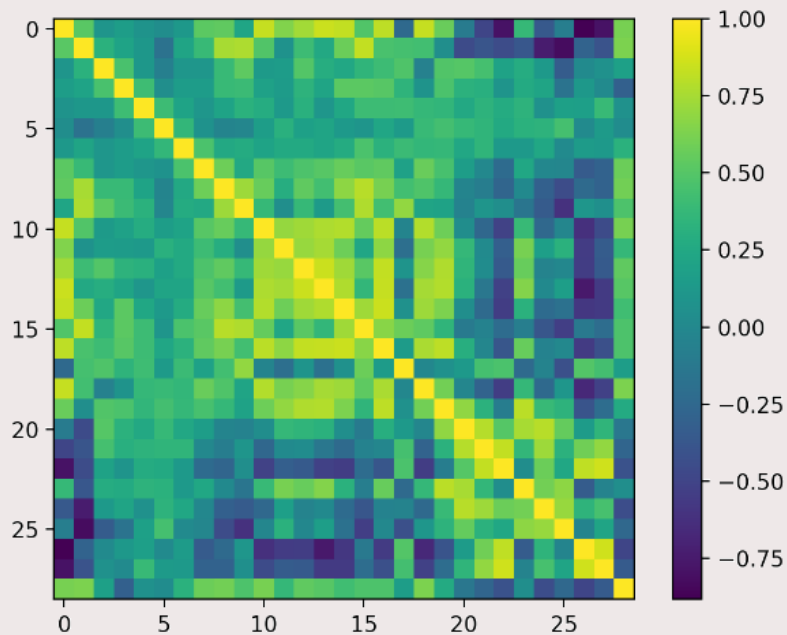
Alexander Van Werde, presented at KU Leuven classical analysis seminar (2023)

# Part I: Context & Related work

## What is matrix concentration?

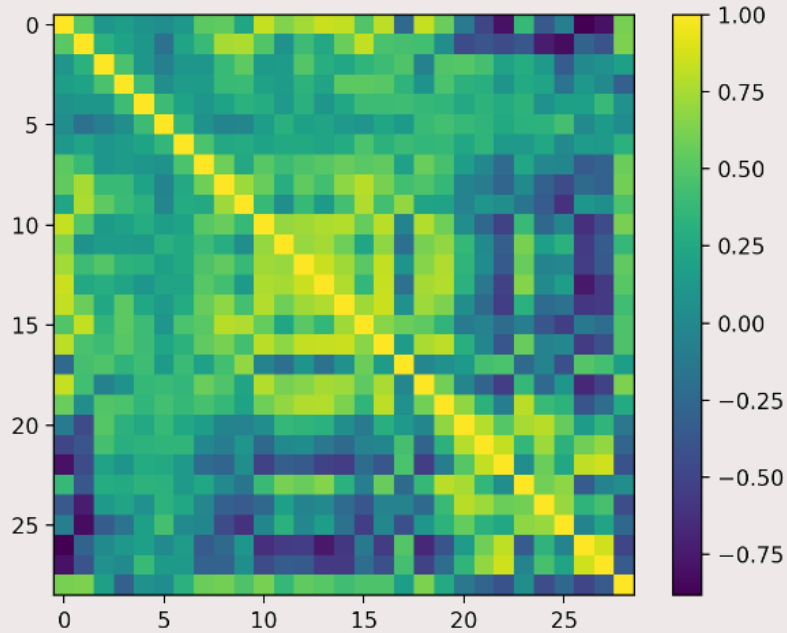
# Applications of matrix concentration

## Time series analysis

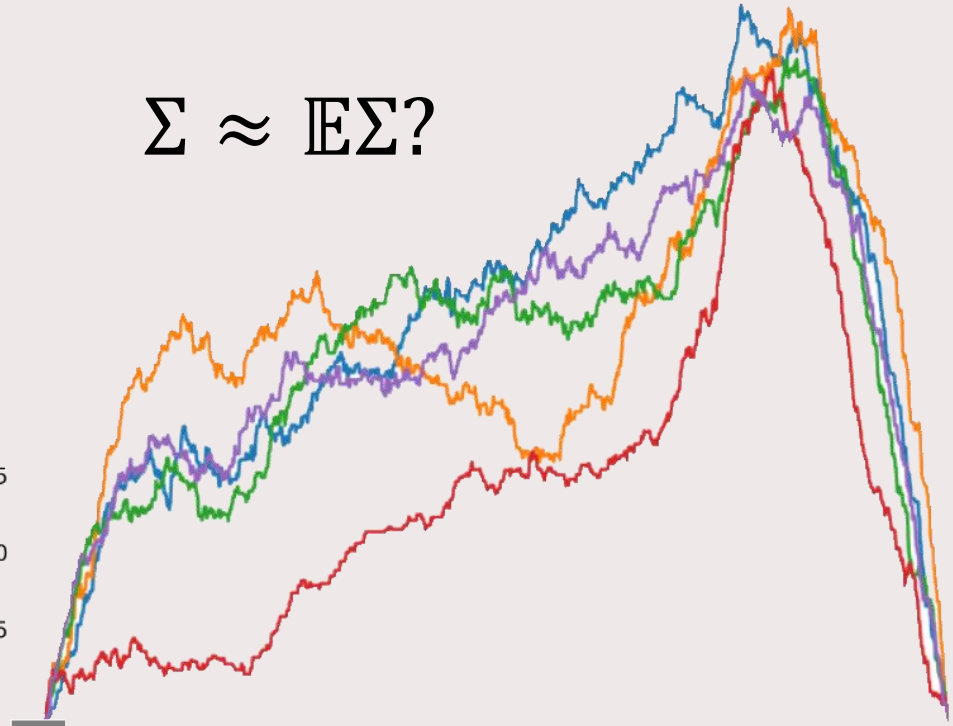


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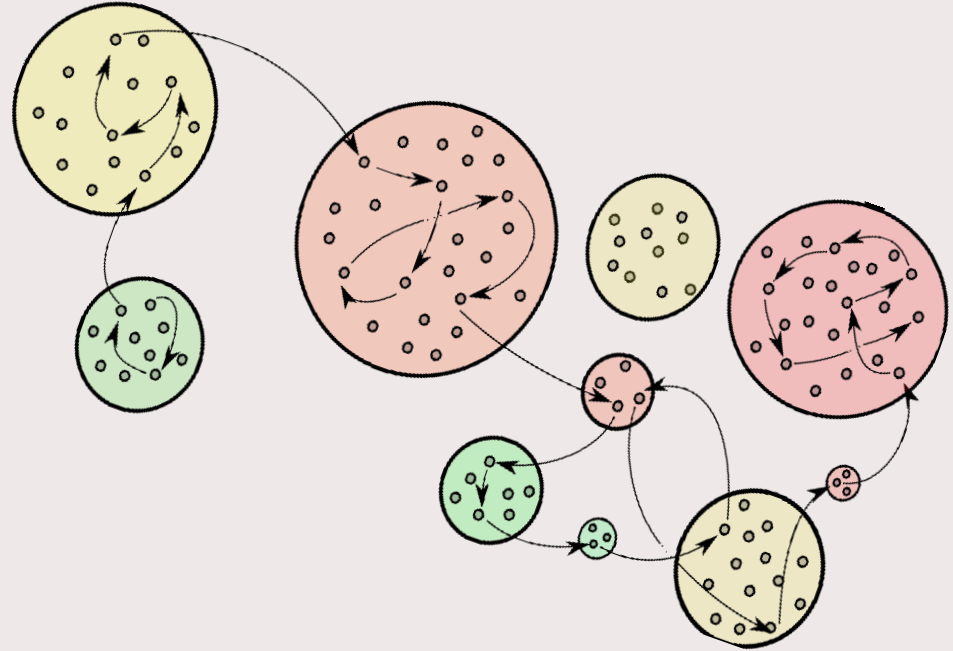
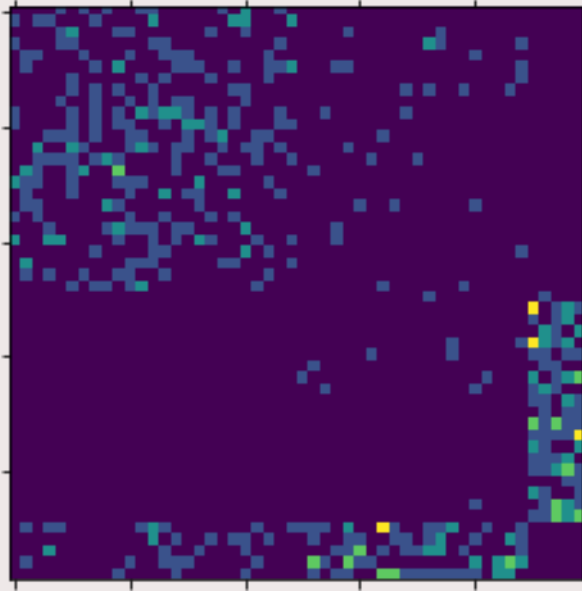


$$\Sigma \approx \mathbb{E}\Sigma?$$



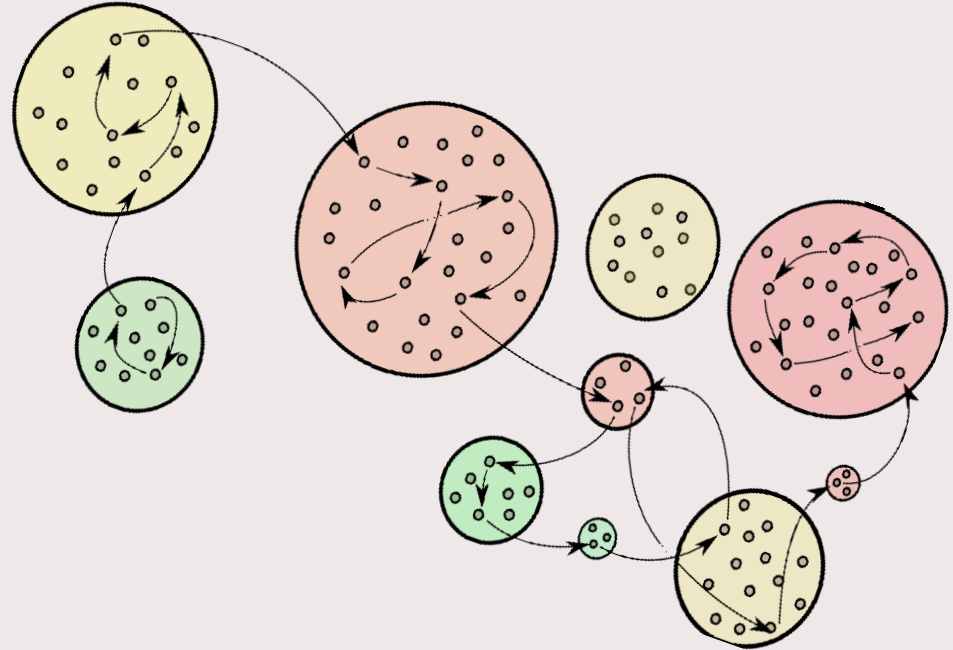
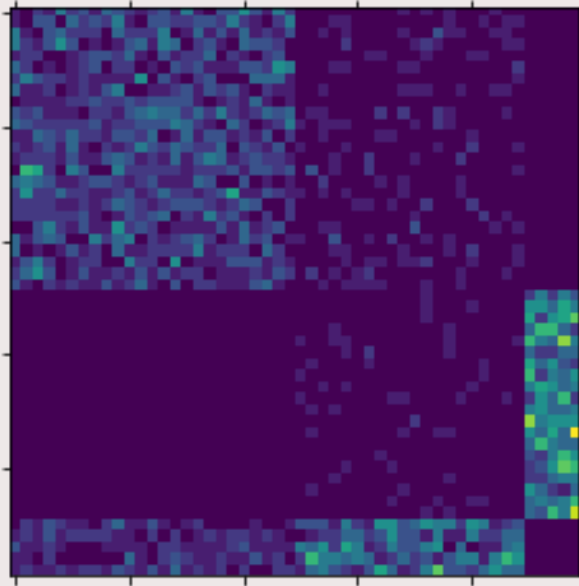
# Applications of matrix concentration

## Community detection



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# Applications of matrix concentration

Time series analysis

Combinatorial optimization

Randomized linear  
algebra

Multivariate statistics

# Compressed sensing

Quantum information theory

Graph sparsification

Community detection

Generalization in machine learning

# Matrix Bernstein inequality

## Theorem. (Tropp, Oliveira)

Let  $\mathbf{X}_1, \dots, \mathbf{X}_n$  be independent self-adjoint  $d \times d$  matrices.

Assume  $\mathbb{E}\mathbf{X}_i = 0$  and  $\|\mathbf{X}_i\| \leq R$ . Denote  $\mathbf{S} := \sum_{i=1}^n \mathbf{X}_i$ .

Then, for some absolute  $c > 0$ , the operator norm satisfies

$$\mathbb{E}\|\mathbf{S}\| \leq c \sqrt{\ln(2d)} \sigma + c \ln(2d) R$$

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### Limitations:

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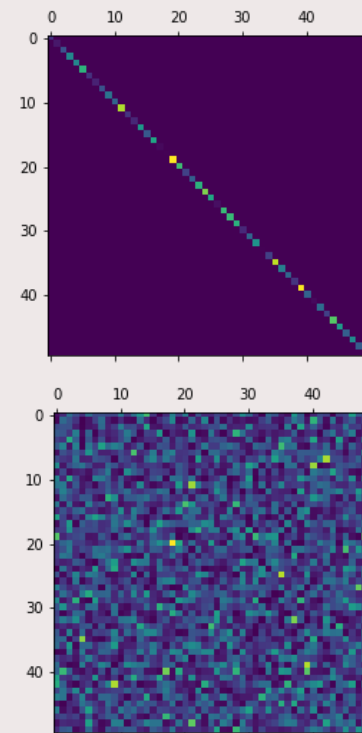
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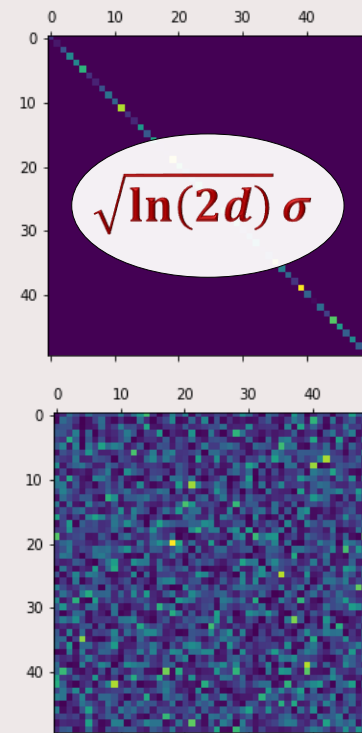
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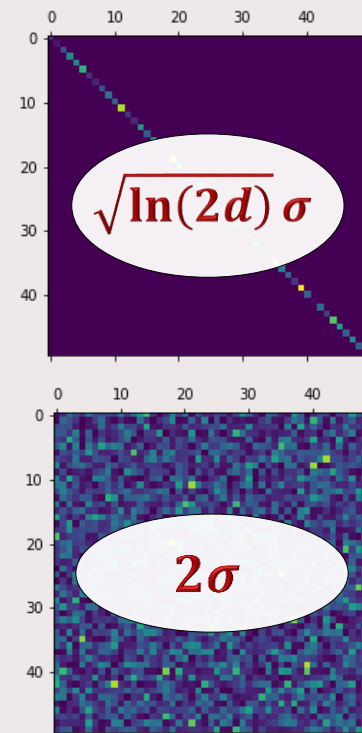
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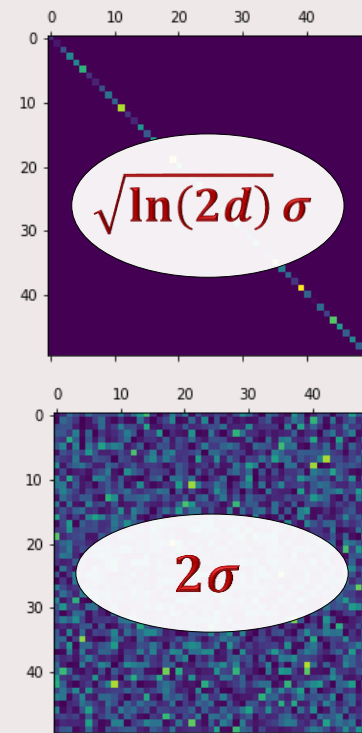
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# Some related literature

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Markovian matrix Bernstein.
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# Part II

## Sharp matrix concentration for Markovian model & matrix series model

# Markovian model and parameters

## Markovian model.

Consider a Markov chain  $Z_1, \dots, Z_n$ .

Generate self-adjoint  $d \times d$  matrices

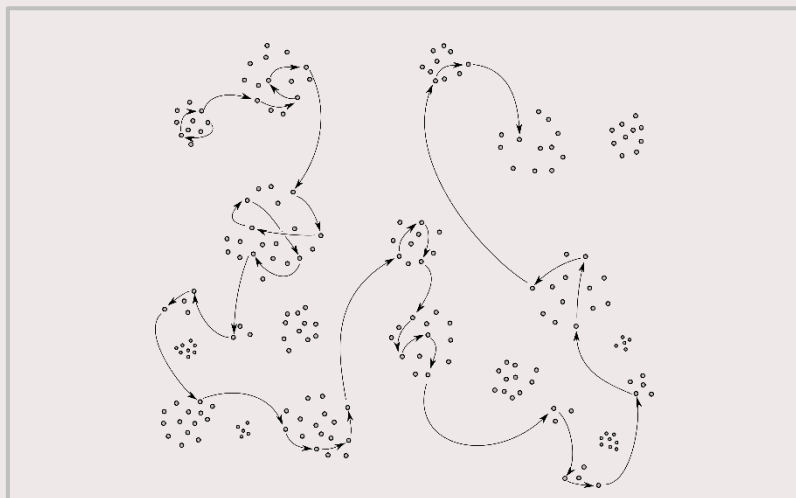
$$\mathbf{X}_i := f_i(Z_i) \quad \text{and} \quad \mathbf{S} := \sum_{i=1}^n \mathbf{X}_i$$

Further, for simplicity, also assume

- Finite state space
- Transition matrix  $P$   
& equilibrium distribution  $\pi$
- $\mathbb{E}\mathbf{X}_i = 0$

## Dependence parameter

$$\Psi := \min\left\{ t \geq 1: \frac{|P_{x,y}^t - \pi_y|}{\pi_y} \leq \frac{1}{4} \quad \forall x, y \right\}$$



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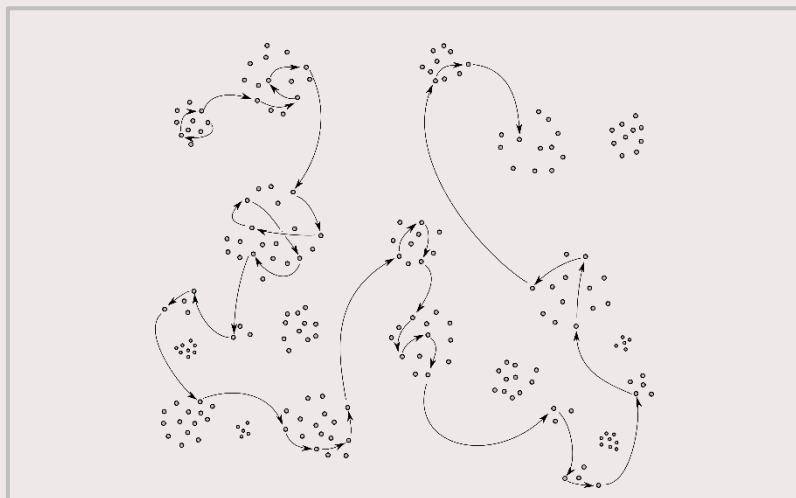
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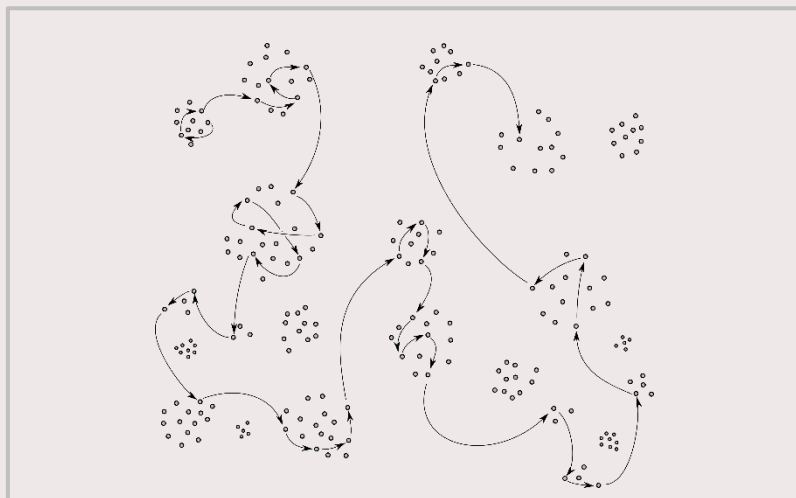
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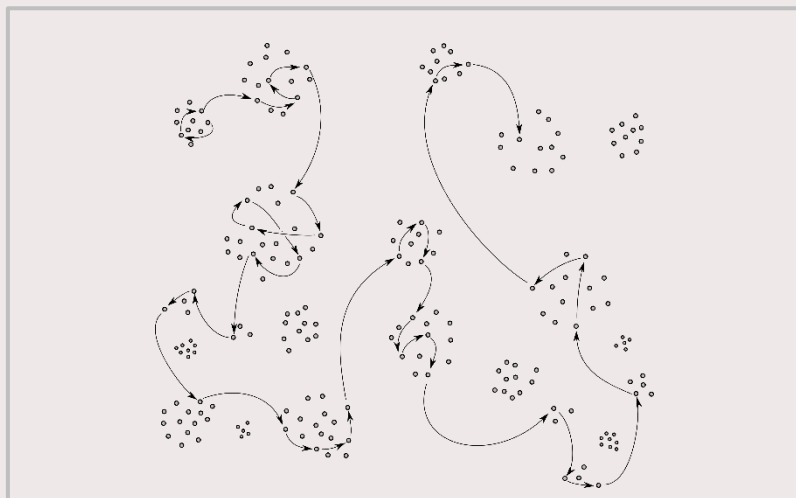
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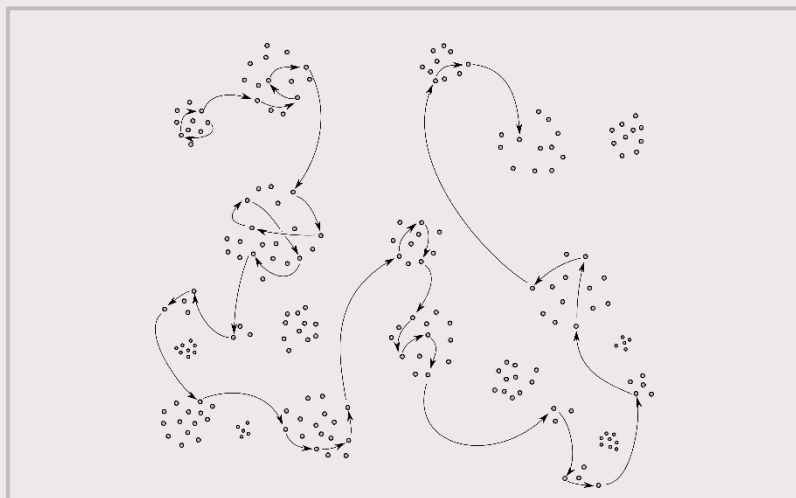
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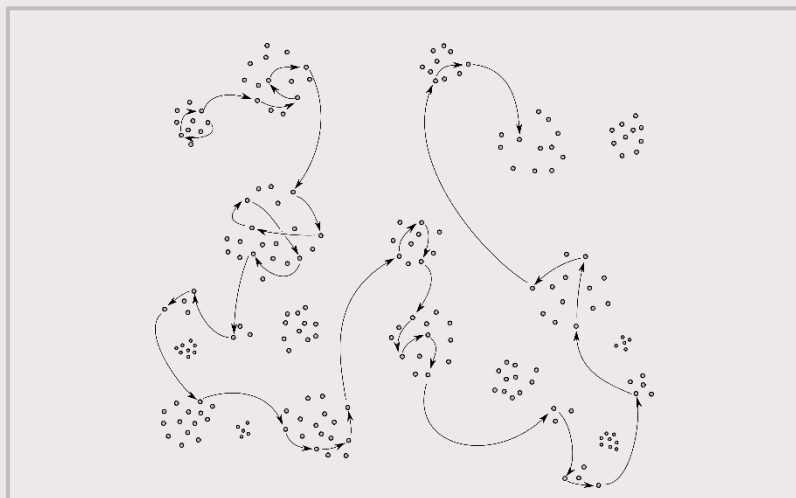
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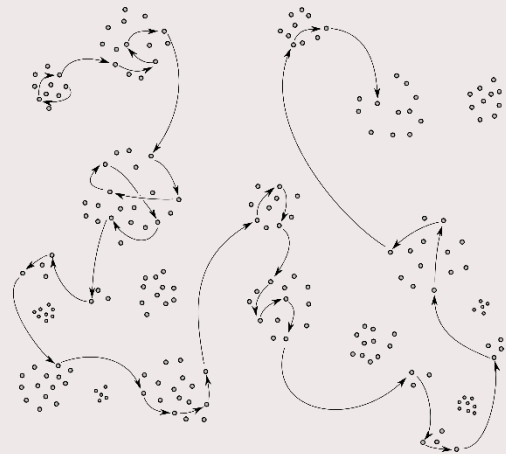
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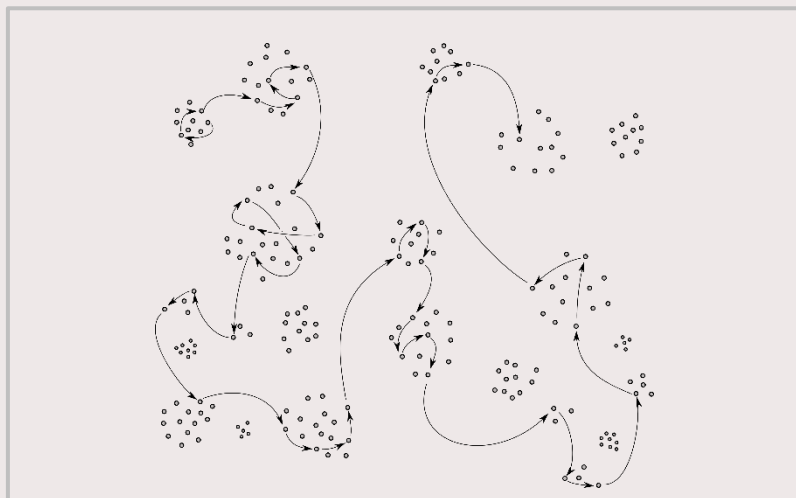
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# Sharp matrix concentration in Markovian model

## Main result.

There exists  $c > 0$  such that for  $0 < \delta \leq 1$

$$\mathbb{E}\|\mathcal{S}\| \leq (1 + \delta)\|\mathcal{S}_{\text{free}}\| + c\mathcal{E}_{d,\delta}$$

where

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$$\Psi := \min\left\{t \geq 1: \frac{|P_{x,y}^t - \pi_y|}{\pi_y} \leq \frac{1}{4} \quad \forall x, y\right\}$$

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# Part III

## Proof sketch: chalkboard



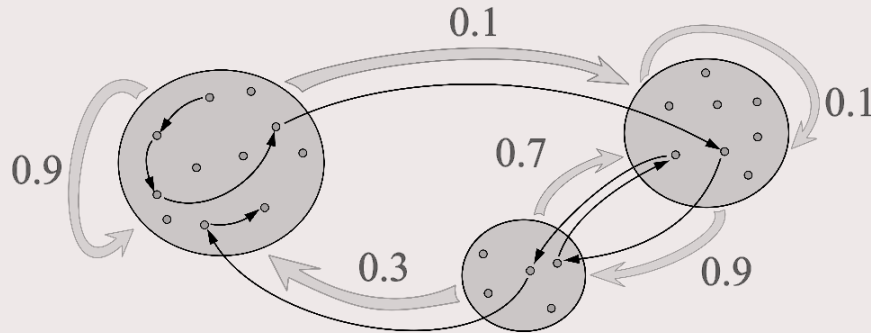
# Part IV

## Illustrative applications

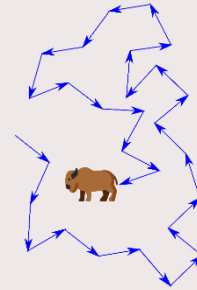
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## Spectral clustering in block Markov chains

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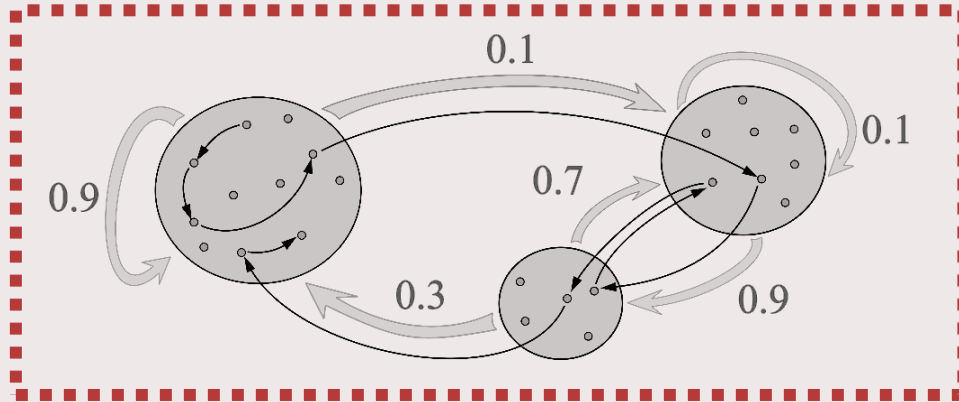
Stock market



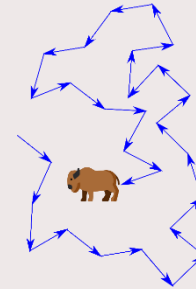
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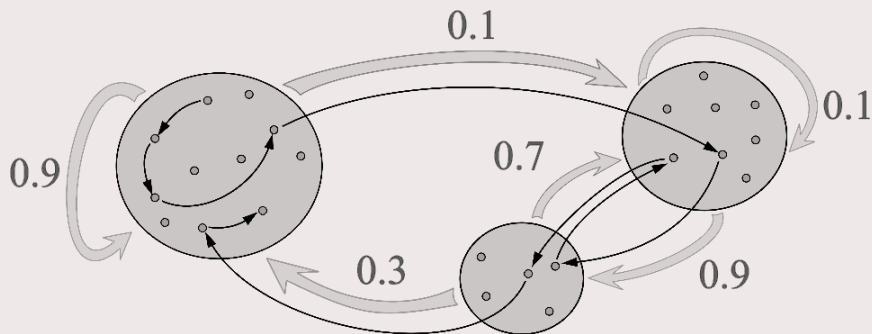
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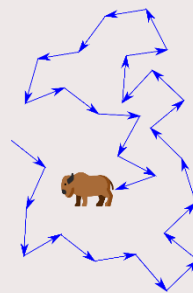
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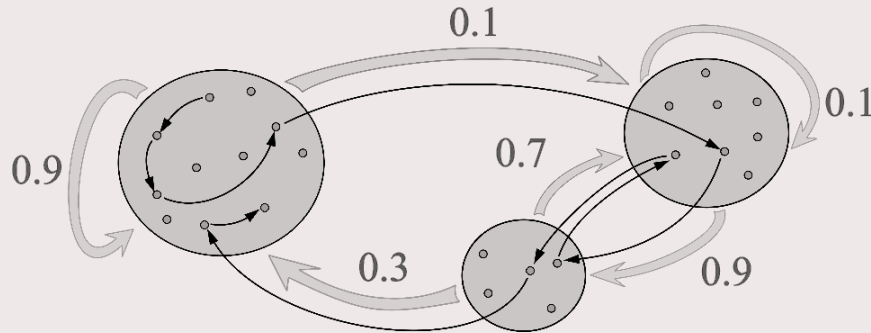
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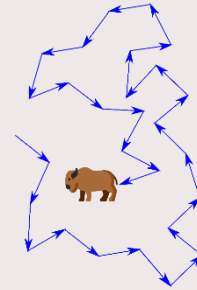
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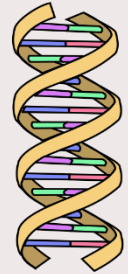
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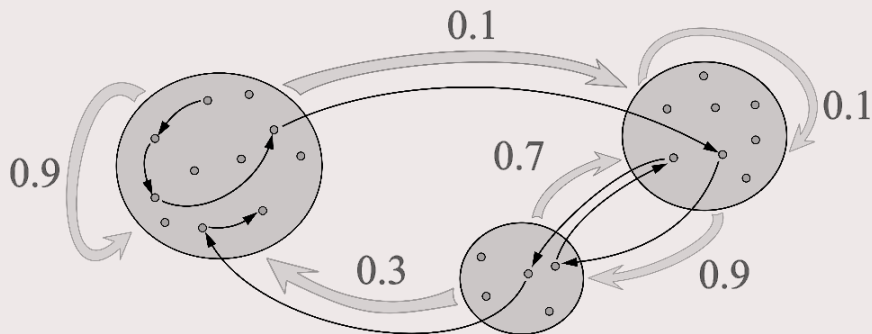
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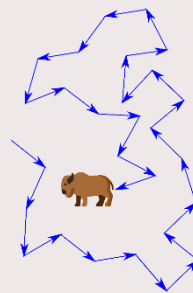
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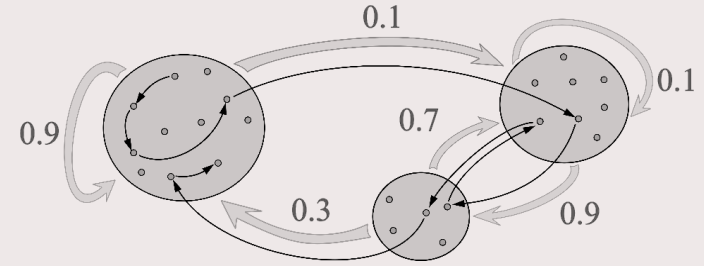


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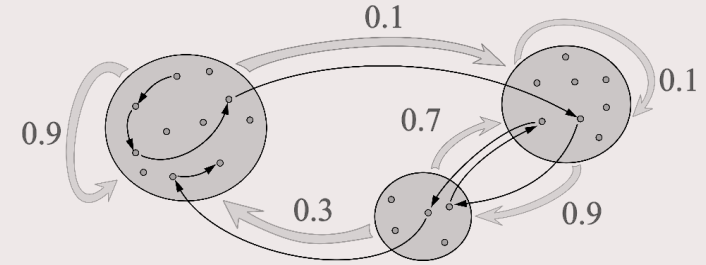


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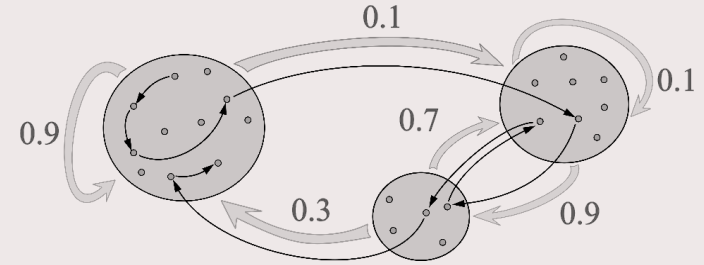


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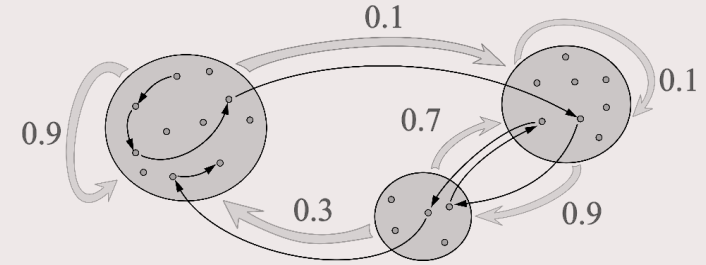


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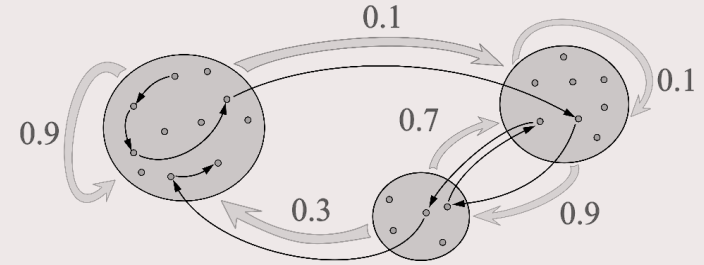


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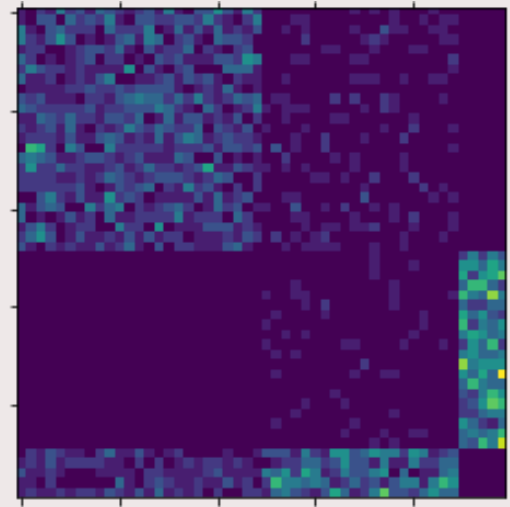
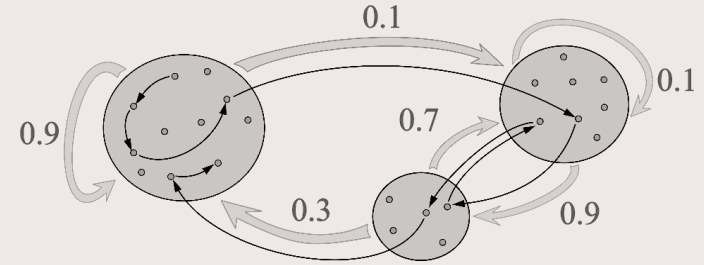
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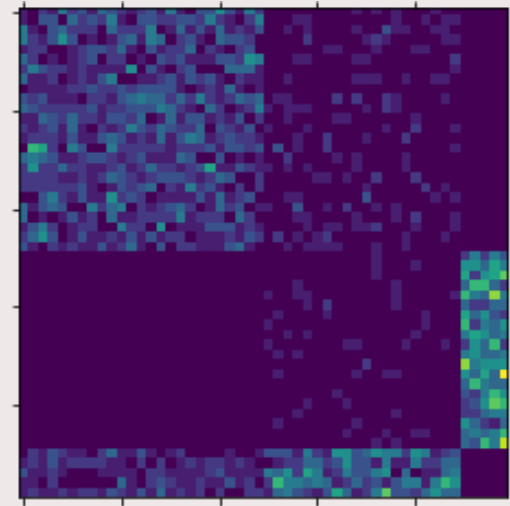
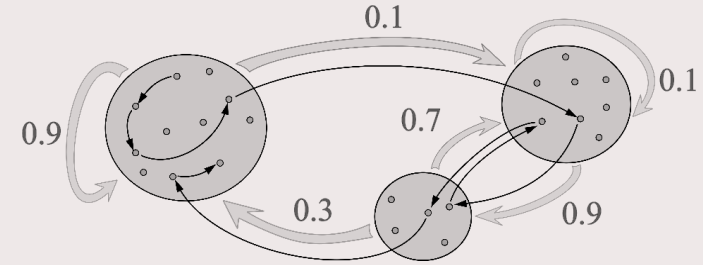
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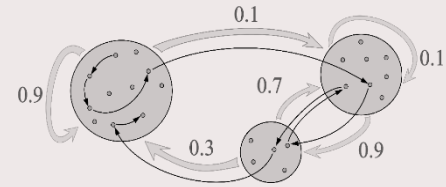
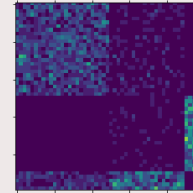
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# Application: Block Markov chains

$$\text{Let } \mathbf{M} := \sqrt{d/n} (\mathbf{N} - \mathbb{E}\mathbf{N})$$



## Corollary.

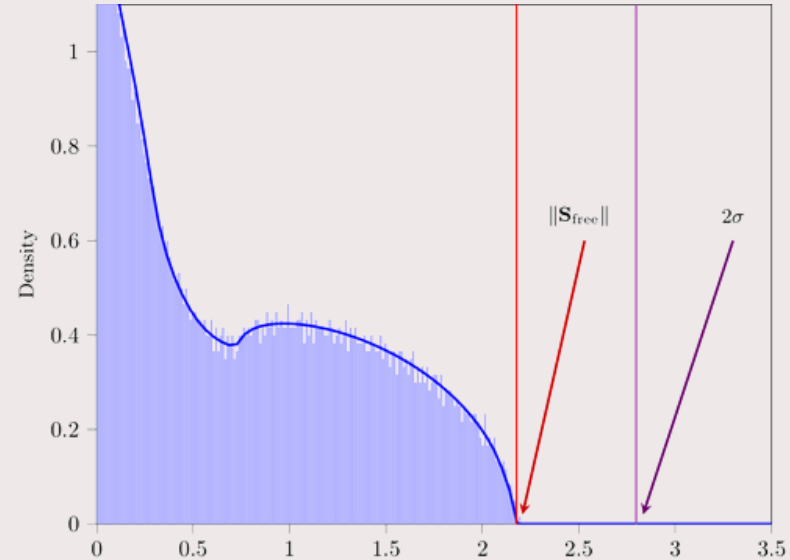
Suppose  $n \gg d \ln(d)^4$  as  $d \rightarrow \infty$ , then

$$\|\mathbf{M}\| \rightarrow \mathcal{M} \quad \text{in probability}$$

where

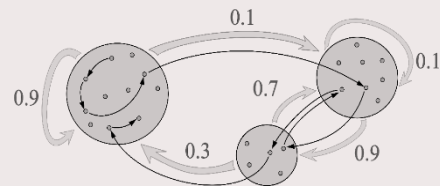
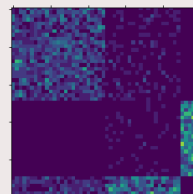
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with explicit coefficients  $c_{ij}$ .



# Application: Block Markov chains

$$\text{Let } \mathbf{M} := \sqrt{d/n} (\mathbf{N} - \mathbb{E}\mathbf{N})$$



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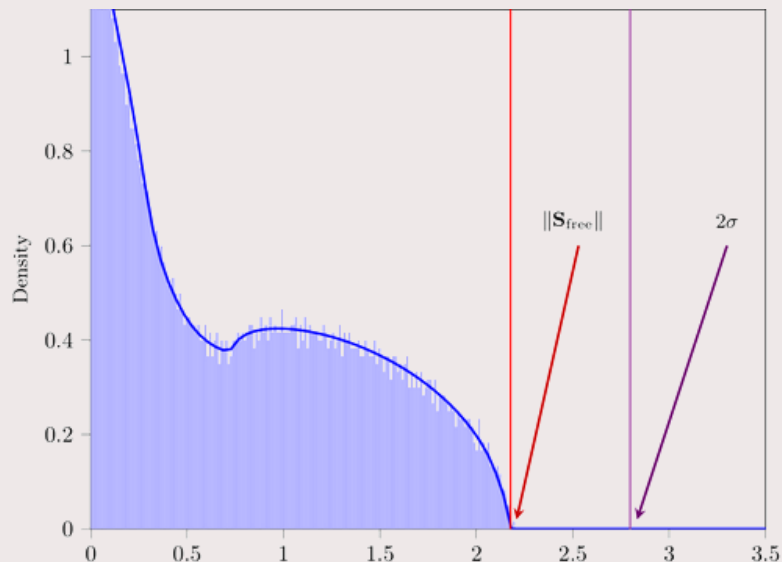
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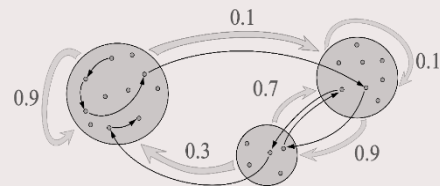
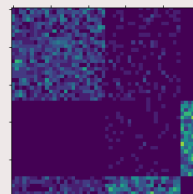
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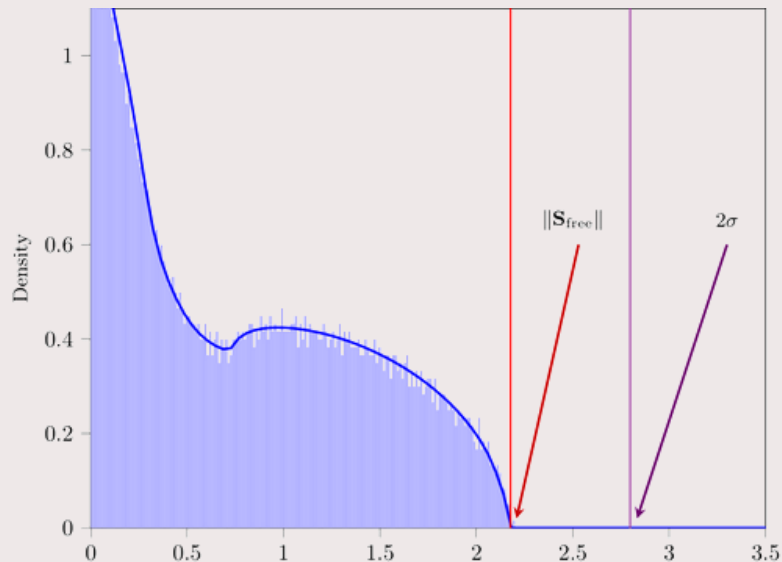
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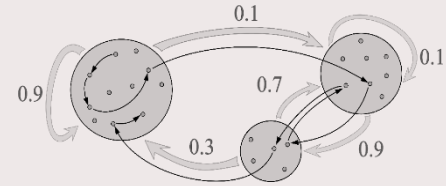
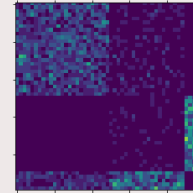
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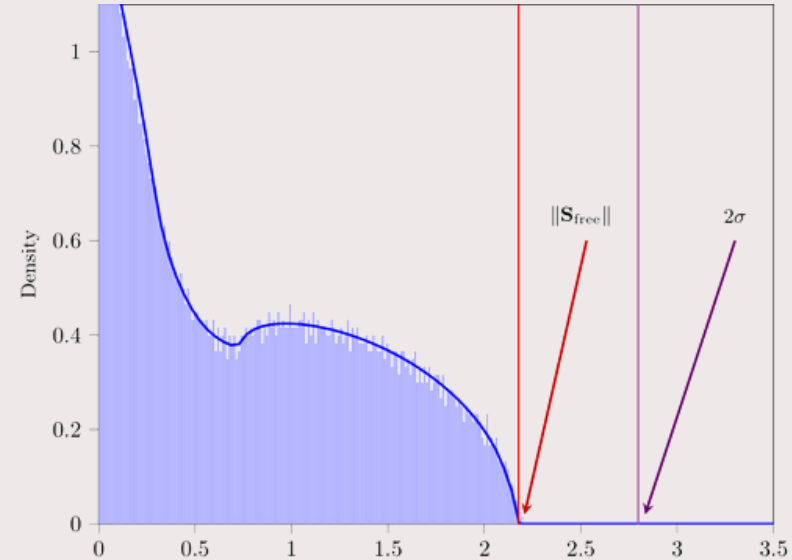
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# Applications of matrix series model

## Heavy-tailed independent entries

Let  $\mathcal{S}$  be a Wigner matrix with independent sub-Weibull entries.

Then, we show that

$$\|\mathcal{S}\| \leq 2\sigma + \varepsilon \text{ with high probability.}$$

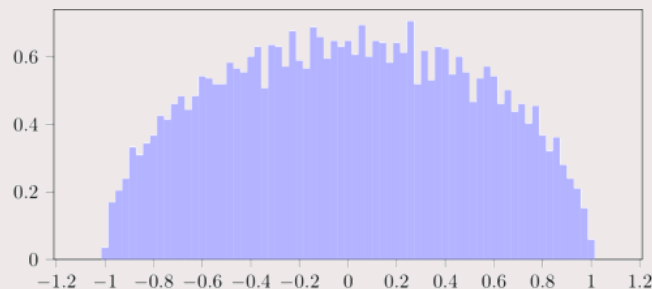
We provide explicit tail bounds.

This is a quantitative non-asymptotic variant on the Bai-Yin theorem.

## Spectra of graphs with dependent edges

Let  $A$  be the adjacency matrix of an Erdős–Rényi random graph with  $m$  edges and  $d$  nodes.

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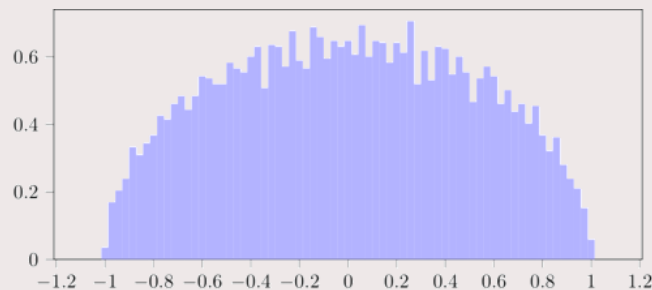
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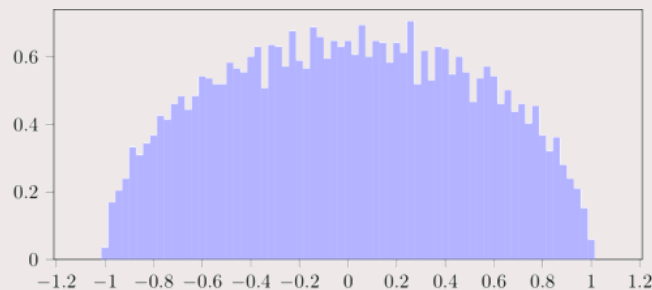
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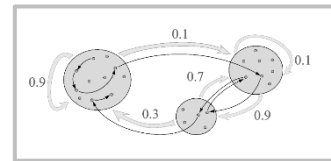
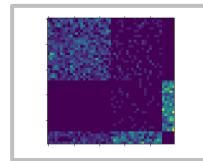
# Summary

We establish *sharp* matrix concentration inequalities for sum of *dependent* random matrices.

Boolean cumulants play a crucial role in the proof. Surprisingly, classical cumulants are insufficient for sharp estimates.

Sharp concentration in applications, such as block Markov chains, would be inaccessible by previous nonasymptotic dependent results.

# Thank you!



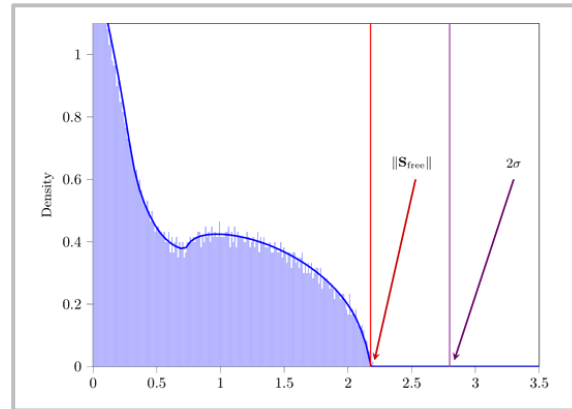
## Main result for Markovian model.

There exists  $c > 0$  such that for any  $0 < \delta \leq 1$

$$\mathbb{E}\|\mathbf{S}\| \leq (1 + \delta)\|\mathbf{S}_{\text{free}}\| + c\mathcal{E}_{d,\delta}$$

where

$$\mathcal{E}_{d,\delta} := v^{1/2}\sigma^{1/2} \log_{1+\delta}(2d)^{3/4} + R^{1/3}\Psi^{2/3}\zeta^{2/3} \log_{1+\delta}(2d)^{2/3} + R\Psi \log_{1+\delta}(2d)$$



## Dependence parameter

$$\Psi := \min\left\{t \geq 1: \frac{|P_{x,y}^t - \pi_y|}{\pi_y} \leq \frac{1}{4}\right\}$$

## Matrix parameters

$$\zeta^2 := \|\mathbb{E}[\sum_{i=1}^n \mathbf{X}_i^2]\|, \quad \sigma^2 := \|\mathbb{E}[\mathbf{S}^2]\|$$
$$v^2 := \|\text{Cov}(\mathbf{S})\|, \quad R \geq \|\mathbf{X}_i\|$$

## Free-probabilistic quantity

$$\|\mathbf{S}_{\text{free}}\| = \inf_{W>0} \lambda_{\max}(W^{-1} + \mathbb{E}[\mathbf{S}W\mathbf{S}]).$$

# Supplementary slides

## Proof sketch

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**Goal.**  $\mathbb{E}[\text{tr } \mathbf{S}^{2p}] \approx \mathbb{E}[\text{tr } \mathbf{G}^{2p}]$  where  $\mathbf{G}$  is a *Gaussian model* for  $\mathbf{S}$

**Interpolate:** For  $t \in [0,1]$  set

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- Expansion with *classical cumulants*.
- Independence-implies-vanishing property reduces combinatorics

We lack independence!

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**Direct modification using classical cumulants gives suboptimal results...**

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