Matrix concentration inequalities with dependent summands and sharp leading-order terms

Alexander Van Werde, presented at SNAPP Seminar (2023)

Joint work with Jaron Sanders

arXiv:2307.11632

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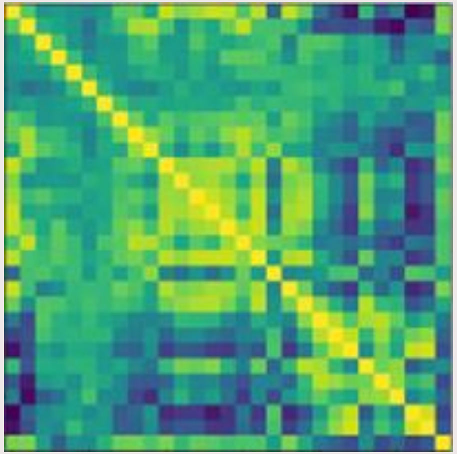

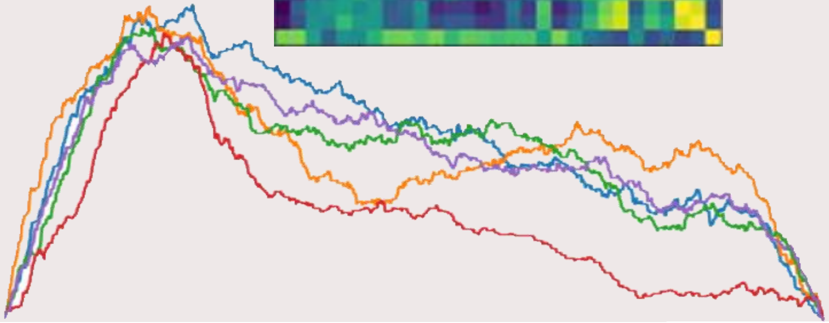

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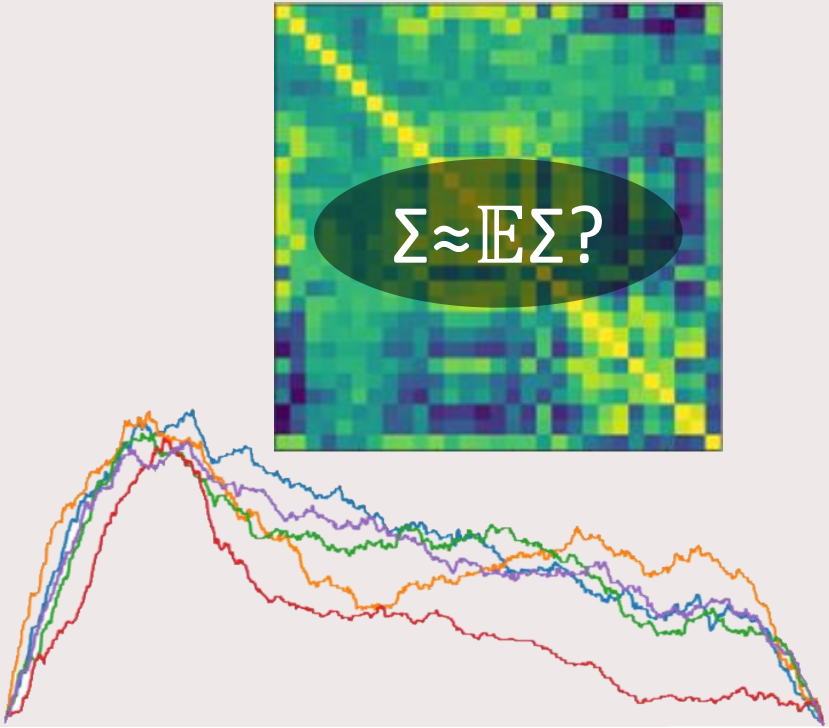

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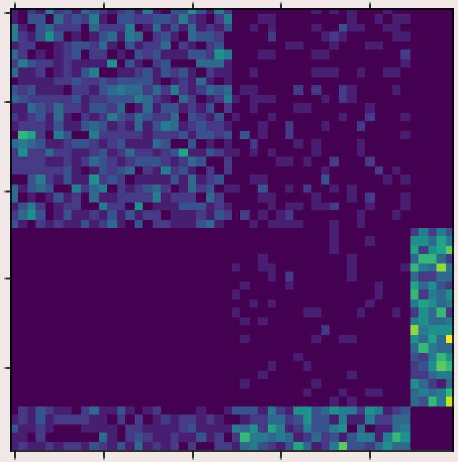

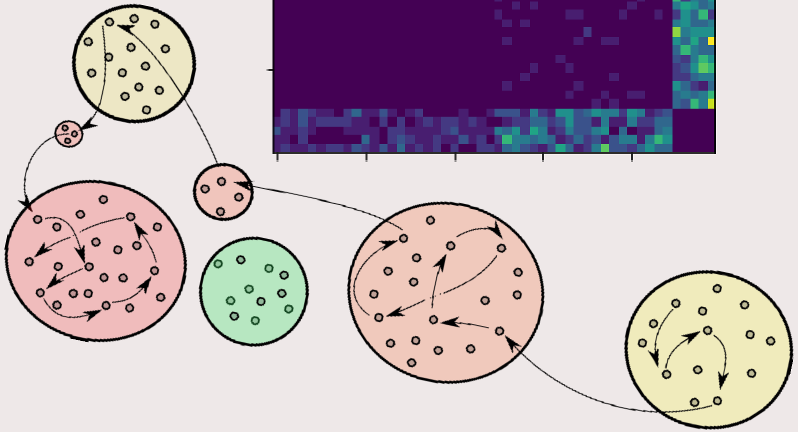

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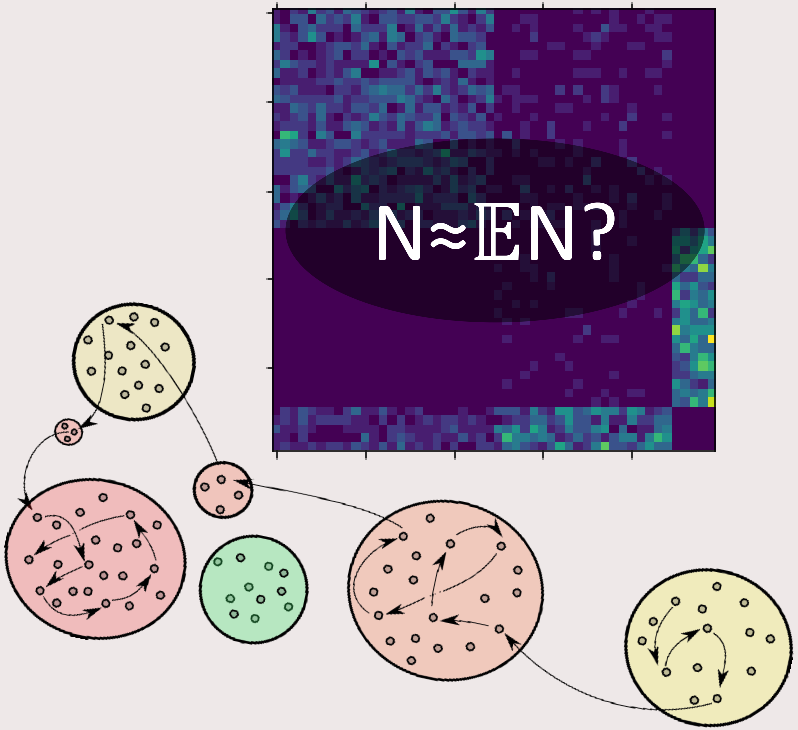

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Convergence rates for Hessians in stochastic gradient descent
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Randomness-efficient sampling methods
- A. Garg, Y.T. Lee, Z. Song, N. Srivastava (2018)

Random walk-based graph embedding methods
- J. Qiu, C. Wang, B. Liao, R. Peng, J. Tang (2020)

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Summary of our contributions

1. Non-asymptotic concentration inequality

$$\|\boldsymbol{M} - \mathbb{E}\boldsymbol{M}\| \leq \|\boldsymbol{M}_{\mathrm{free}}\| + \varepsilon$$

   for matrices $\boldsymbol{M}$ generated by a dependent process.

2. Asymptotically sharp and easy to use in applications!
   - Block Markov chains
   - Random matrices with heavy-tailed entries
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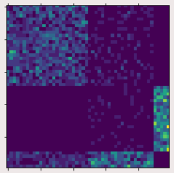
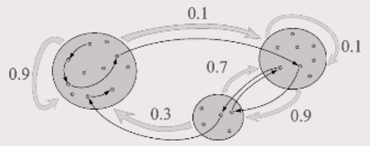
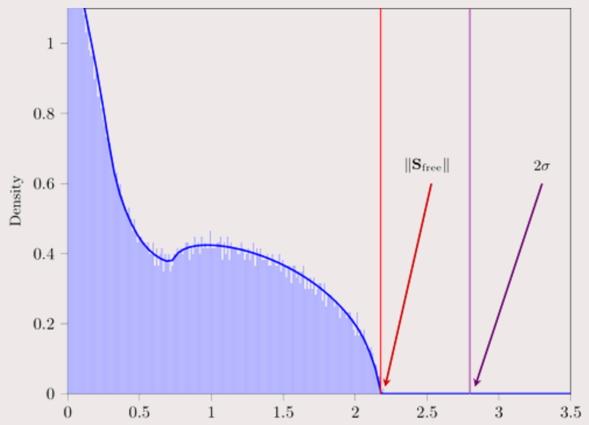
3. Proof techniques to deal with dependence.
   (Boolean cumulants, change-of-measure, combinatorial estimates, …)







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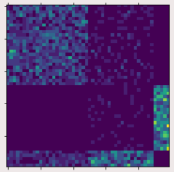

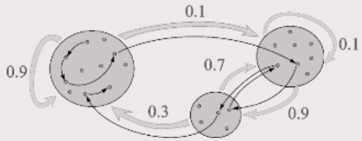

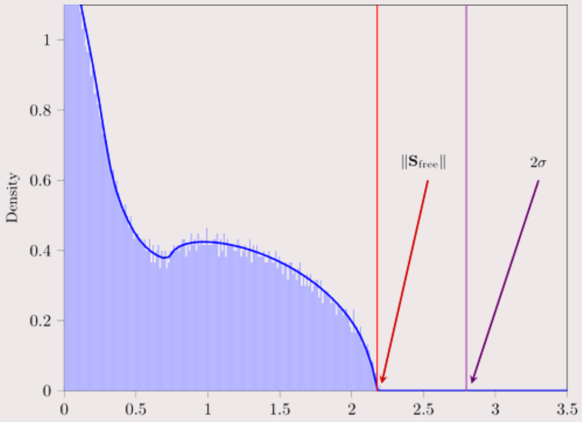

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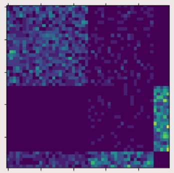

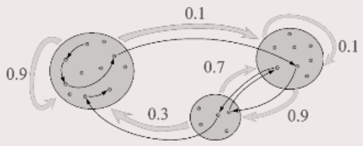

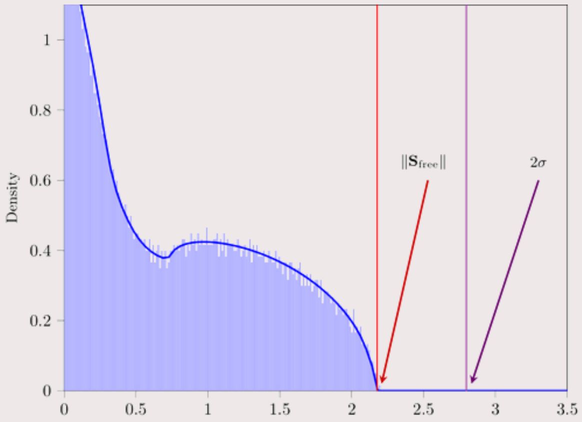

Context: suboptimality of classical matrix concentration

Matrix Khintchine (Lust–Piquard)

There exist absolute constants $c, C > 0$ so that any self-adjoint Gaussian $d \times d$ matrix $\boldsymbol{M}$ satisfies

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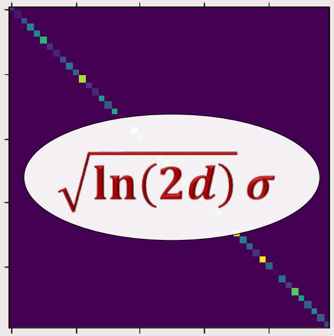

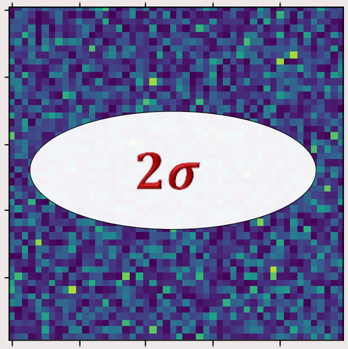

Some related literature

Matrix concentration with non-Gaussian dependent summands

- **R.I. Oliveira (2009):**
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Sharp matrix concentration
for Markovian model & matrix series model

Markovian model and parameters

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Consider a Markov chain $Z_1, \ldots, Z_n$.

Generate self-adjoint $d \times d$ matrices

$$\boldsymbol{X}_i := f_i(Z_i) \quad \text{and} \quad \boldsymbol{M} := \sum_{i=1}^{n} \boldsymbol{X}_i$$

Further, for simplicity, also assume

- Finite state space
- Transition matrix P
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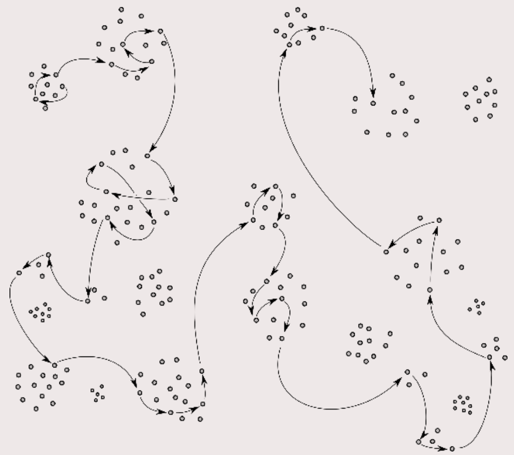
- $\mathbb{E}\boldsymbol{X}_i = 0$

Dependence parameter

$$\Psi := \min\{\, t \geq 1 : \frac{|P_{x,y}^t - \pi_y|}{\pi_y} \leq \frac{1}{4} \; \forall x, y\}$$



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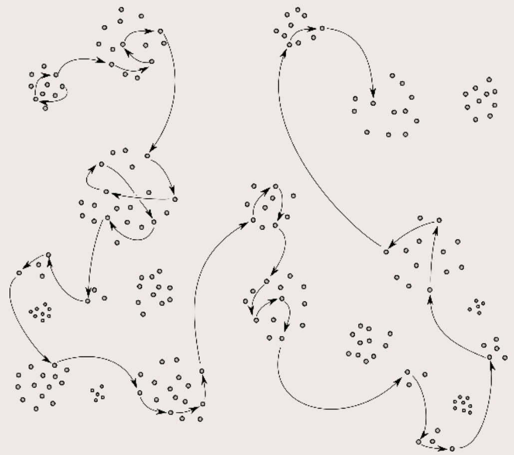

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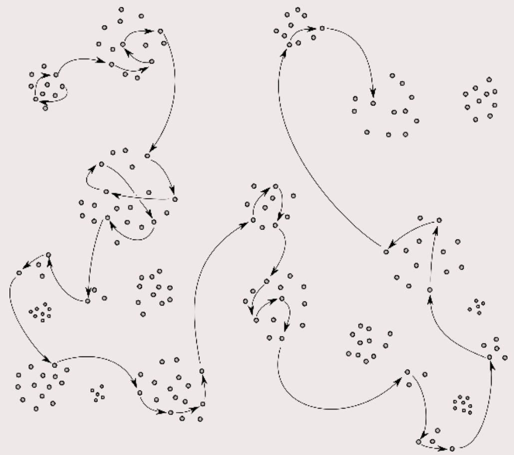

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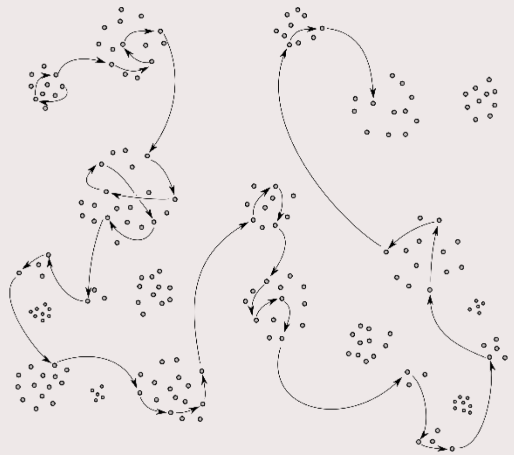

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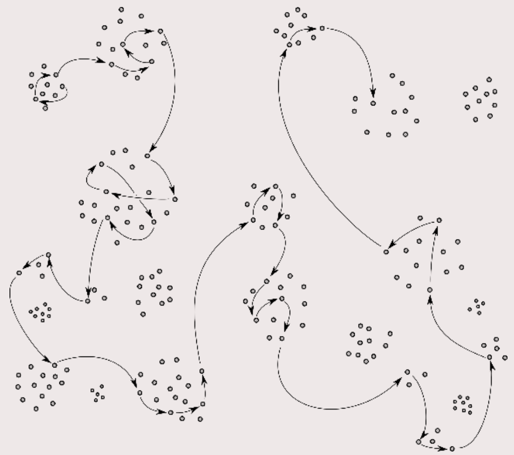

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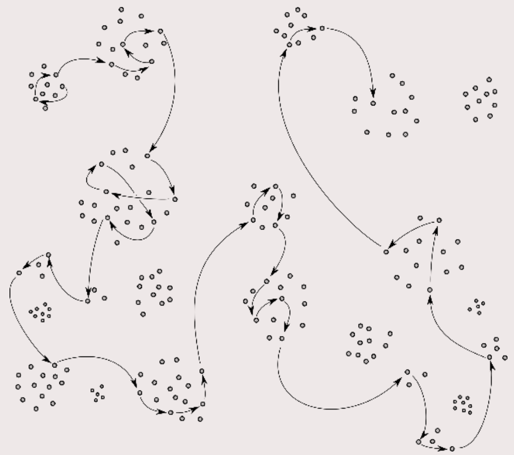

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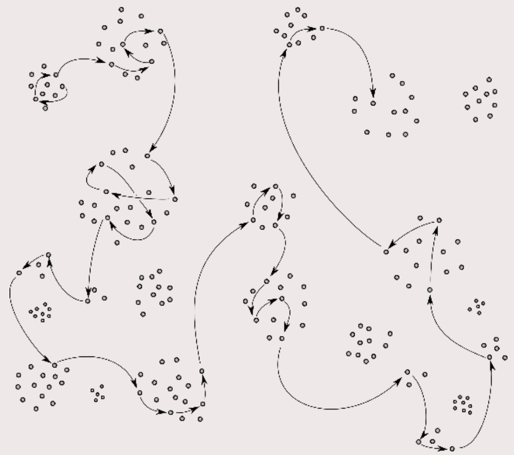

Sharp matrix concentration in Markovian model

Main result.

There exists $c > 0$ such that for $0 < \delta \leq 1$

$$\mathbb{E}\|\boldsymbol{M}\| \leq (1+\delta)\|\boldsymbol{M}_{\text{free}}\| + c\mathcal{E}_{d,\delta}$$

where

$$\mathcal{E}_{d,\delta} := v^{1/2}\sigma^{1/2}\log_{1+\delta}(2d)^{3/4} + R^{1/3}\Psi^{2/3}\varsigma^{2/3}\log_{1+\delta}(2d)^{2/3} + R\,\Psi\log_{1+\delta}(2d)$$

Dependence parameter

$$\Psi := \min\{\, t \geq 1 : \frac{|P^t_{x,y} - \pi_y|}{\pi_y} \leq \frac{1}{4}\ \forall x, y\}$$

Free-probabilistic quantity

$$\|\boldsymbol{M}_{\text{free}}\| = \inf_{\boldsymbol{W} \succ \boldsymbol{0}} \lambda_{\max}\left(\boldsymbol{W}^{-1} + \mathbb{E}[\boldsymbol{M}\boldsymbol{W}\boldsymbol{M}]\right).$$

Matrix parameters

$$\varsigma^2 := \left\|\mathbb{E}\left[\textstyle\sum_{i=1}^n \boldsymbol{X}_i^2\right]\right\|, \quad \sigma^2 := \|\mathbb{E}[\boldsymbol{M}^2]\|$$

$$v^2 := \|\mathrm{Cov}(\boldsymbol{M})\|, \qquad R \geq \|\boldsymbol{X}_i\|.$$

Sharp matrix concentration in Markovian model

Main result.

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TU/e

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$$\sigma \leq \|\boldsymbol{M}_{\text{free}}\| \leq 2\sigma$$

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In many relevant models

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Matrix series model.

We show a similar result for a *matrix series model*.

There, summands are of the form $\boldsymbol{X}_i := Y_i \boldsymbol{A}_i$ for deterministic $\boldsymbol{A}_i$.

Proof sketch

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Goal. $\mathbb{E}[\operatorname{tr} \boldsymbol{M}^{2p}] \approx \mathbb{E}[\operatorname{tr} \mathbf{G}^{2p}]$ where $\boldsymbol{G}$ is a *Gaussian model* for $\boldsymbol{M}$

Interpolate: For $t \in [0,1]$ set $\boldsymbol{M}(t) := \sqrt{t}\boldsymbol{M} + \sqrt{1-t}\boldsymbol{G}$ and show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[\operatorname{tr} \boldsymbol{M}(t)^{2p}] \leq \epsilon(t)$$

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We lack independence!

Direct modification using classical cumulants gives suboptimal results…

Summary

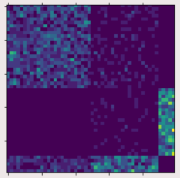
arXiv:2307.11632

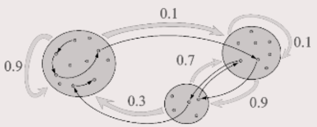
1. We achieve sharp concentration inequalities in a dependent setting

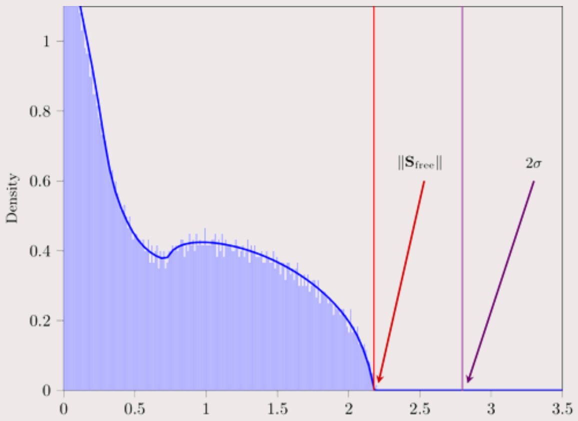
$$\|\boldsymbol{M} - \mathbb{E}\boldsymbol{M}\| \leq \|\boldsymbol{M}_{\text{free}}\| + \varepsilon$$

2. Surprisingly, classical cumulants are insufficient. We use the (more obscure) Boolean cumulants instead.

3. Applications such as clustering in Markov chains and random graphs may be found in the paper.







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