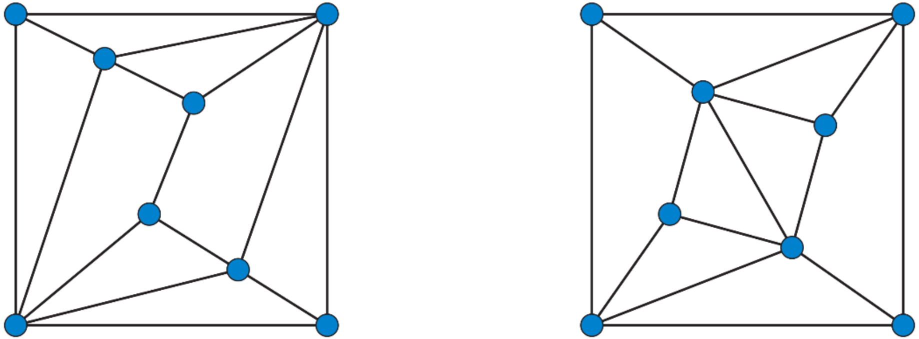
Cokernel statistics of walk matrices

Towards generalized spectral determinacy of random graphs



Alexander Van Werde, TU/e lunch seminar in Combinatorial Optimization (2024)

Preprint available at arXiv:2401.12655

TU/e EINDHOVEN UNIVERSITY OF TECHNOLOGY

Can *all* information about a graph be recovered from its spectrum?

Some history: spectral determinacy of graphs

1957 (Collatz and Sinogowitz)

No, non-isomorphic graphs can have the same adjacency spectrum.

1973 (Schwenk)

Almost all trees are cospectral!

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"It is an open question whether almost all graphs are characterized by their characteristic polynomials. It is not even clear if we should seek to prove this, or to disprove it."

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Conjecture: almost all graphs are determined by spectrum.

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2023 (Koval and Kwan)

At least $\exp(cn)$ graphs are determined by spectrum.

We lack flexible proof techniques…

Some history: generalized spectral determinacy

1980 (Johnson and Newman)

"It is our view, however, that to some extent these examples are algebraic accidents due to the interpretation of the formal symbols 0 and 1 as real numbers."

Definition/Theorem. (Generalized cospectral)

The following are equivalent:

1. For all $x, y \in \mathbb{R}$, one has $\mathrm{spec}\left(\mathrm{A}_{\mathrm{G}}^{x,y}\right) = \mathrm{spec}\left(\mathrm{A}_{\mathrm{H}}^{x,y}\right)$.
   Here, $A_G^{x,y}$ is a variant on adjacency matrix with $1 \to x$ and $0 \to y$.
2. The graphs G and H have the same spectrum and the complement graphs have the same spectrum.

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Conjecture (Wang): Satisfied with constant probability! 😮

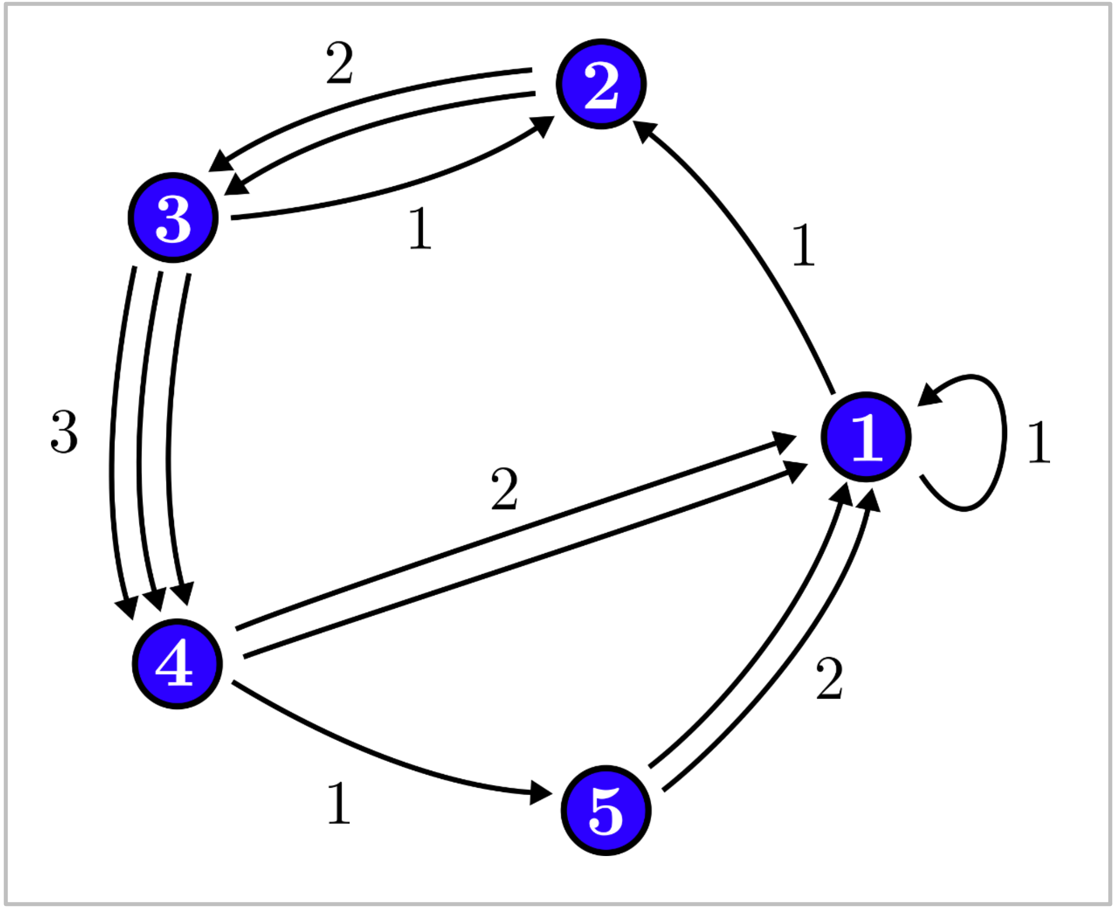
Sufficient condition for generalized spectral determinacy

Definition (Walk matrix)

Given an integer matrix $\boldsymbol{X} \in \mathbb{Z}^{n\times n}$, we let

$$\boldsymbol{W} := [e, \boldsymbol{X}e, , \boldsymbol{X}^2 e, \dots, \boldsymbol{X}^{n-1}e]$$

where $e = (1, \dots, 1)^{\mathrm{T}}$.

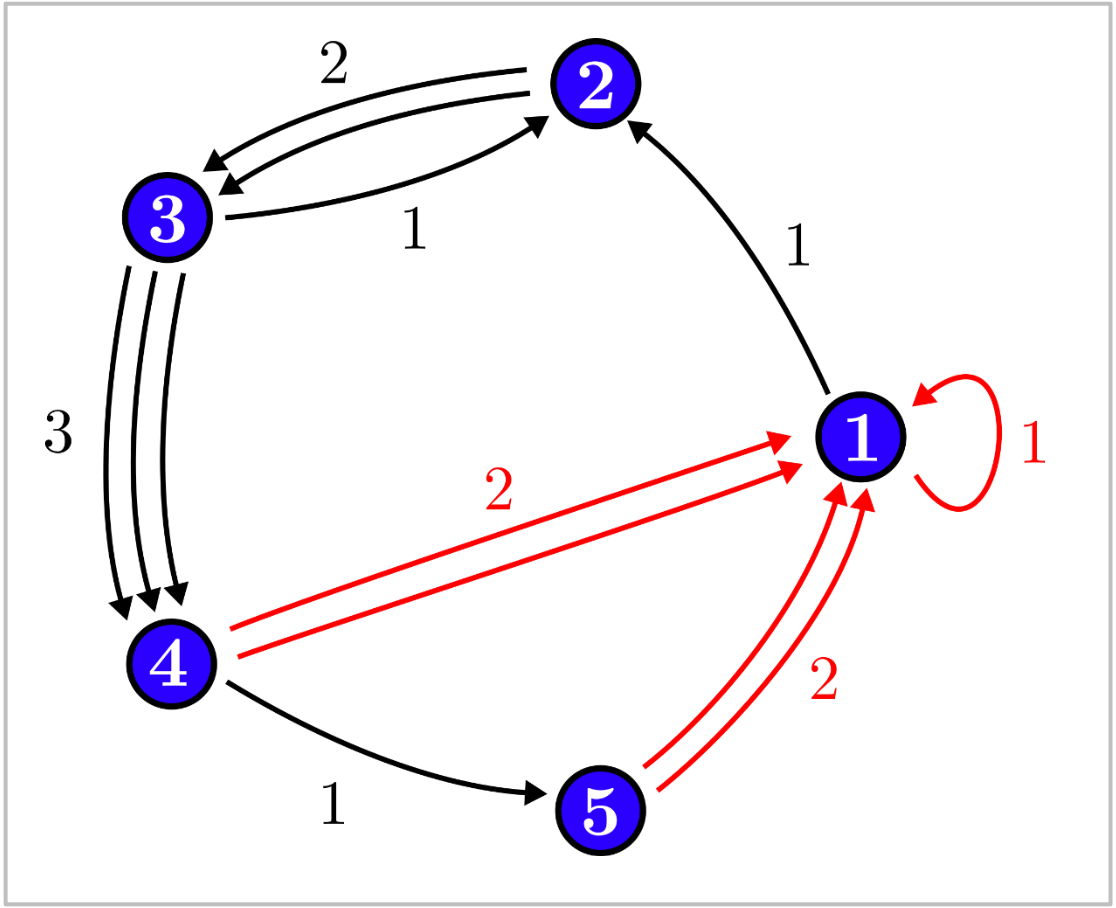


Definition:

$$\boldsymbol{W} \coloneqq [e, \boldsymbol{X}e, \dots, \boldsymbol{X}^{n-1}e]$$

Interpret $\boldsymbol{X}_{i,j}$ as edge multiplicity.

Then, $\boldsymbol{W}_{i,j}$ counts walks of length $j-1$ ending in i.



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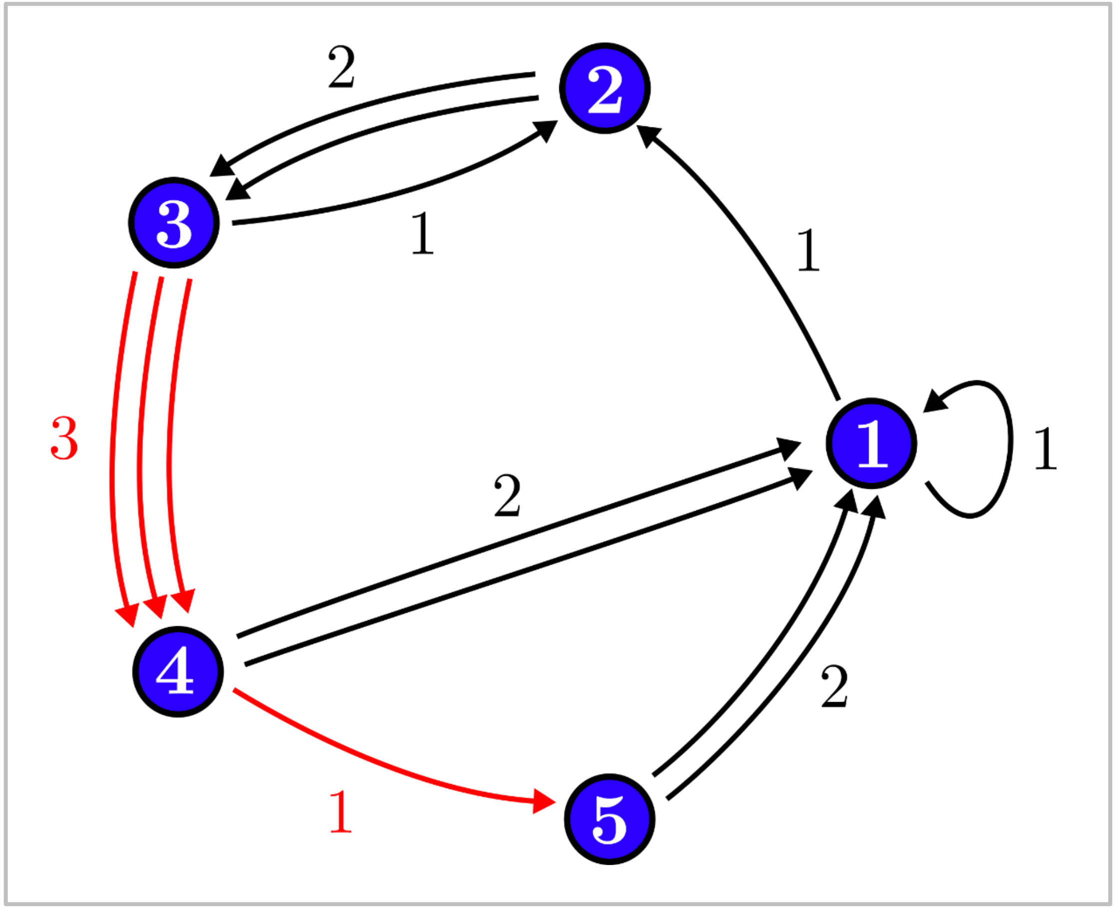
$$W_{1,2} = 2 + 2 + 1 = 5$$



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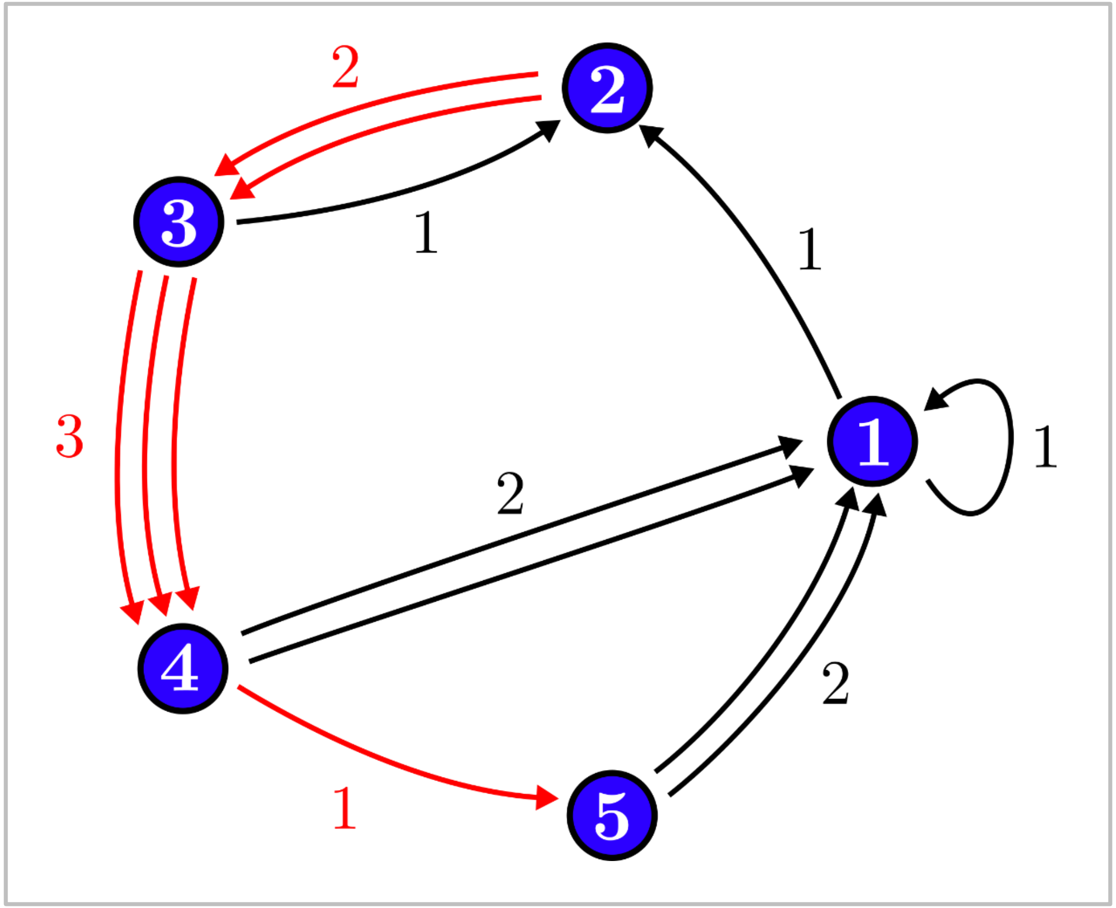

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$$W_{5,4} = 2 \cdot 3 \cdot 1 = 6$$

Sufficient condition for generalized spectral determinacy

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Notation

Given an Abelian group G and a prime power p^m, let $G_{p^m} \coloneqq G/p^m G$.

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Theorem. (Wang 2017; see also Qui, Wang, and Zhang 2023)

Consider a simple graph G and set $\boldsymbol{X} \coloneqq \boldsymbol{A}_G$. Assume that $\operatorname{coker}(\mathbf{W})_{2^2} \cong (\mathbb{Z}/2\mathbb{Z})^{\lfloor n/2 \rfloor}$ and $\operatorname{coker}(\mathbf{W})_{p^2} \in \{0, \mathbb{Z}/p\mathbb{Z}\}$ for odd primes p.

Then, G is determined by generalized spectrum.

Suppose $\boldsymbol{X}$ is random.

How can we study the distribution of $\mathbf{coker}(\boldsymbol{W})$?

Results

Disclaimer.

For technical reasons, all results assume that $\boldsymbol{X}$ has independent entries.

This implies that we can not (yet) deal with the adjacency matrices of *simple* random graphs: those have dependent entries due to the symmetry constraint $\boldsymbol{X} = \boldsymbol{X}^{\mathrm{T}}$.

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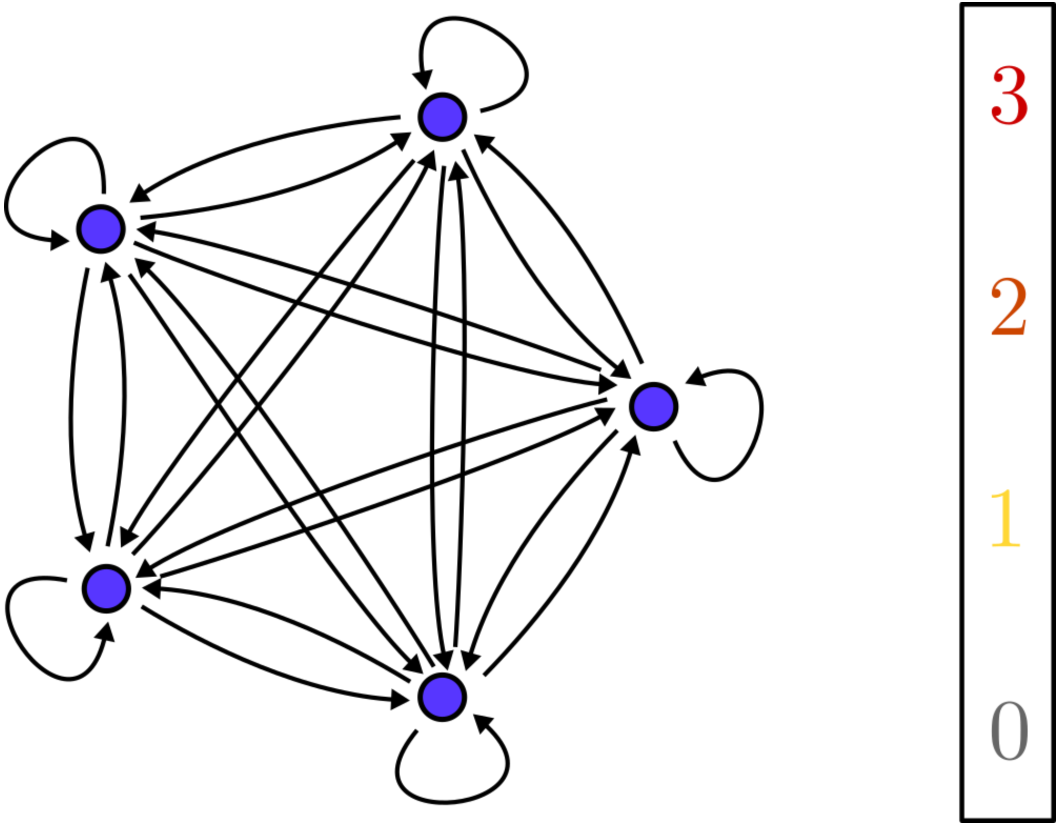
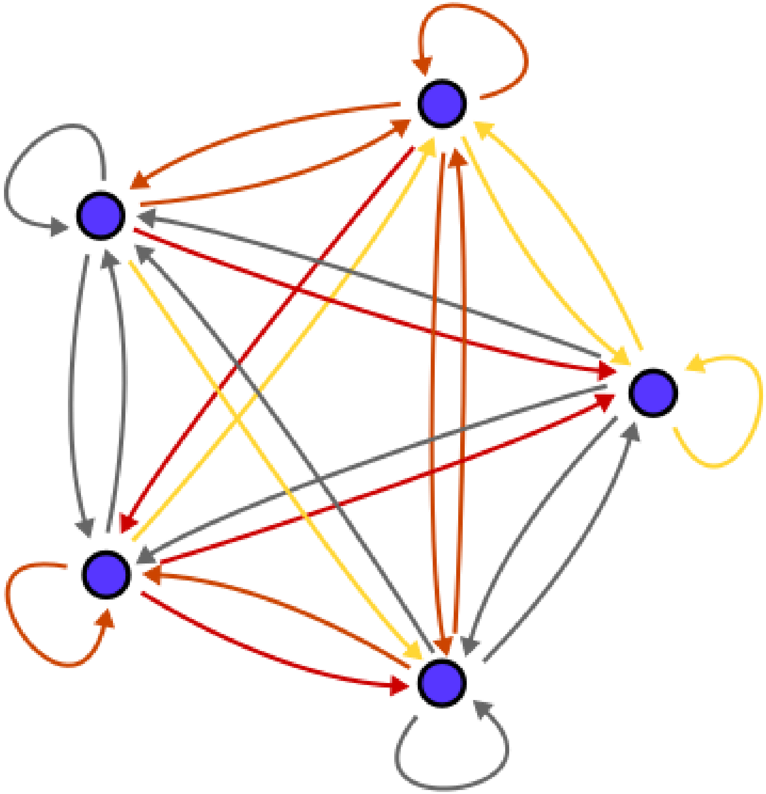
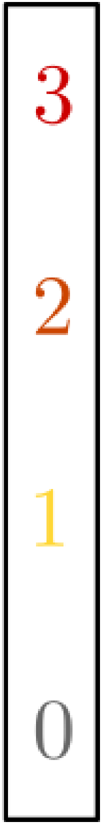
Assumption 1st result

$\boldsymbol{X}$ has independent $\mathrm{Unif}\{0,1,\dots,p^m - 1\}$-distributed entries for some prime p and $m \geq 1$.


3
2
1
0

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Results

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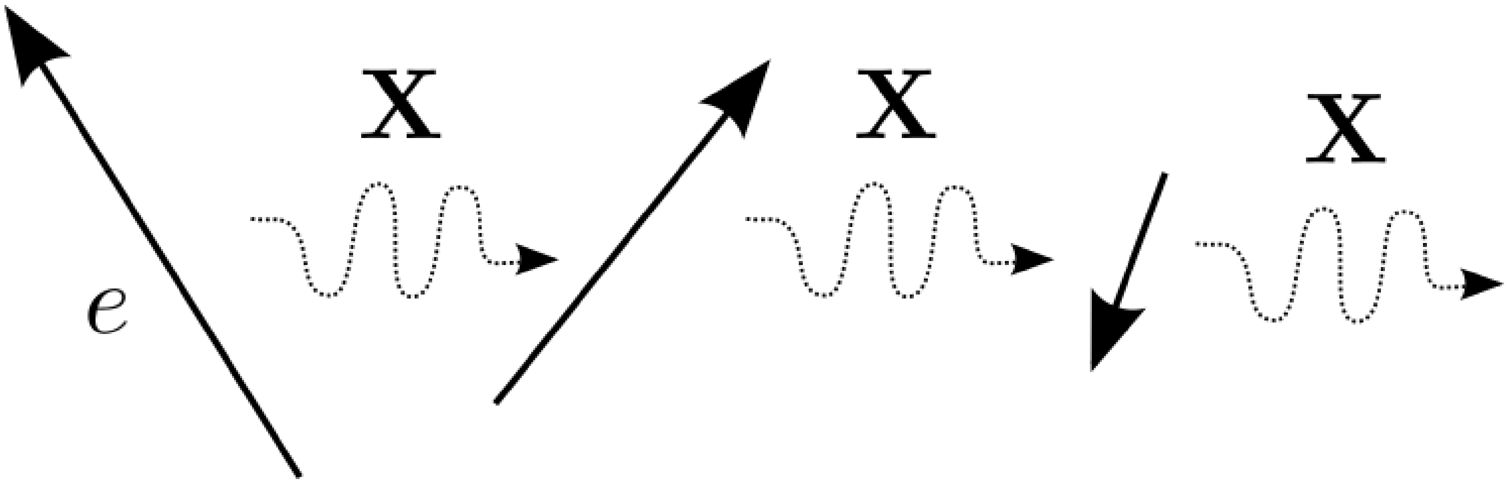
Theorem 1.

Fix $\ell \geq 1$, let $0 = \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_d \leq m$, and set $i_0 := \#\{i \leq \ell : \lambda_i = m\}$.
Denote $\delta_1 := \lambda_\ell - \lambda_{\ell-1}$, $\delta_2 := \lambda_{\ell-1} - \lambda_{\ell-2}, \dots, \delta_\ell := \lambda_1 - \lambda_0$.

Then,

$$\lim_{n\to\infty} \mathbb{P}\left(\mathrm{coker}(\boldsymbol{W})_{p^m} \cong \bigoplus_{i=1}^{\ell} \frac{\mathbb{Z}}{p^{\lambda_i}\mathbb{Z}}\right) = \prod_{i=i_0}^{\infty}\left(1 - p^{-(i+1)}\right) \prod_{j=1}^{\ell} p^{-j\delta_j}$$

Proof idea



Proof idea

There is dependence!

Observe that if

$$\boldsymbol{X}^j e \in \operatorname{span}_{\mathbb{Z}}\left(e, \boldsymbol{X}e, \dots, \boldsymbol{X}^{j-1}e\right) + p^k\mathbb{Z}^n$$

then also

$$\boldsymbol{X}^{j+1} e \in \operatorname{span}_{\mathbb{Z}}\left(e, \boldsymbol{X}e, \dots, \boldsymbol{X}^{j-1}e, \boldsymbol{X}^j e\right) + p^k\mathbb{Z}^n.$$

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Key observation. (Informally)

Aside from the obstruction above, there is independence.

Interpretable proof!

Sadly, the technique is not robust.

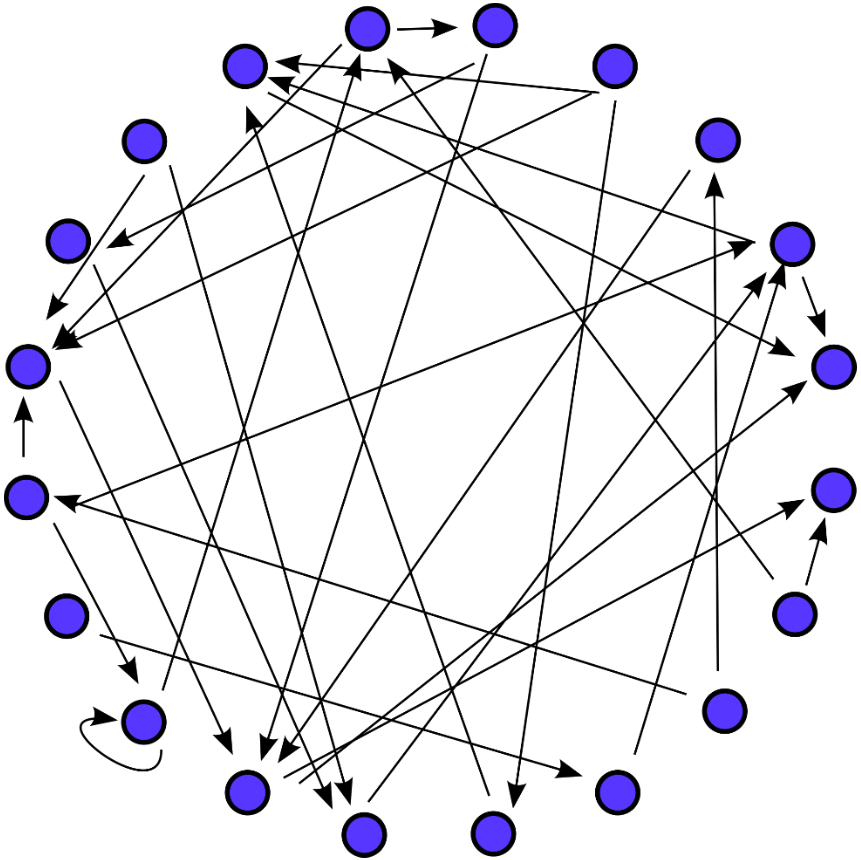
How can we study *unweighted* directed graphs?

Results

Assumption simplified 2nd result

$\boldsymbol{X}$ has independent $\{0,1\}$-valued entries with $\mathbb{P}(\boldsymbol{X}_{i,j} = 1) \leq \mathbb{P}(\boldsymbol{X}_{i,j} = 0)$ and

$$\mathbb{P}\left(\boldsymbol{X}_{i,j} = 1\right) \gg \ln(n)/n$$



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Notation.

Equip $\operatorname{coker}(\boldsymbol{W})$ with $\mathbb{Z}[x]$-module structure from the action of $\boldsymbol{X}$.

Given $Q \in \mathbb{Z}[x]$ and a $\mathbb{Z}[x]$-module N, let

$$N_{p^m,Q(x)} \coloneqq \frac{N}{p^m N + Q(x)N}$$

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Theorem 2. (Simplified)

Fix $Q(x) \in \mathbb{Z}[x]$ and a finite collection of primes $\mathcal{P}$. We show

1. Asymptotic independence for different primes:

$$\lim_{n\to\infty} \mathbb{P}\big(\mathrm{coker}(\boldsymbol{W})_{p^m,Q(x)} \cong N_{p^m,Q(x)}, \forall p \in \mathcal{P}\big)$$
$$= \prod_{p\in\mathcal{P}} \lim_{n\to\infty} \mathbb{P}\big(\mathrm{coker}(\boldsymbol{W})_{p^m,Q(x)} \cong N_{p^m,Q(x)}\big).$$

2. An explicit formula for the limiting law of $\mathrm{coker}(\mathbf{W})_{p^m,Q(x)}$.

Robust proof technique:

category-theoretic moment method.

Category-theoretic moment method

Definition. (Category-theoretic moment)

Consider a ring R, a deterministic R-module N, and a random R-module Y.

Then, the N-moment of Y is $\mathbb{E}[\#\mathrm{Sur}_R(Y, N)]$.

Theorem. (Sawin and Wood, 2022)

Consider a random R-module Y and a sequence of R-modules Y_n.

Then, under certain conditions, to prove that $Y_n \to Y$ in distribution it suffices to show that

$$\lim_{n\to\infty} \mathbb{E}[\#\mathrm{Sur}_R(Y_n, N)] = \mathbb{E}[\#\mathrm{Sur}_R(Y, N)]$$

for every fixed finite R-module N.

Category-theoretic moment method

We show that $\mathbb{E}\left[\#\mathrm{Sur}_{\mathbb{Z}[x]}(\mathrm{coker}(\boldsymbol{W}), N)\right] = (\#N)^{-1}$ for every finite $\mathbb{Z}[x]$-module N.

Related problems were studied by e.g., Nguyen and Wood (2022) and Cheong and Yu (2023).

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Proof sketch.

Using that group morphism $F: \mathbb{Z}^n \to N$ descends to $\mathbb{Z}[x]$-module morphism from $\mathrm{coker}(\boldsymbol{W})$ if and only if $F(e) = 0$ and $F\boldsymbol{X} = xF$,

$$\mathbb{E}[\#\mathrm{Sur}_{\mathbb{Z}[x]}(\mathrm{coker}(\boldsymbol{W}), N)] = \sum_{F \in \mathrm{Sur}_{\mathbb{Z}}(\mathbb{Z}^n, N):\, F(e)=0} \mathbb{P}(F\boldsymbol{X} = xF).$$

There are approximately $(\#N)^{n-1}$ summands since

$$\#\{F \in \mathrm{Sur}_{\mathbb{Z}}(\mathbb{Z}^n, N):\ F(e) = 0\} \approx \#\{F \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^n, N):\ F(e) = 0\}.$$

For typical F, one has $\mathbb{P}(F\boldsymbol{X} = xF) \approx (\#N)^{-n}$.

Future work

Extension to simple graphs.

The prime 2 behaves very different.

Odd primes are qualitatively similar. Numerics suggest a small quantitative difference.

Conjecture.

If $\boldsymbol{X}$ has independent $\{0,1\}$-valued entries with $\mathbb{P}(\boldsymbol{X}_{i,j} = 1) \leq \mathbb{P}(\boldsymbol{X}_{i,j} = 0)$ and $\mathbb{P}\left(\boldsymbol{X}_{i,j} = 1\right) \gg \ln(n)/n$.

Then,

$$\lim_{n\to\infty} \mathbb{P}\left(\operatorname{coker}(\boldsymbol{W})_{p^2} \in \{0, \mathbb{Z}/p\mathbb{Z}\} \text{ for every odd prime } p\right)$$
$$= \prod_{p \text{ odd}}(1 - p^{-2}) \prod_{i=1}^{\infty}(1 - p^{-(i+1)})$$

Thank you!

Key reference related to this talk are as follows:

Generalized spectral determinacy:

W. Wang and C.-X. Xu. *A sufficient condition for a family of graphs being determined by their generalized spectra*. European Journal of Combinatorics, 2006.

W. Wang. *A simple arithmetic criterion for graphs being determined by their generalized spectra.* Journal of Combinatorial Theory, Series B, 2017.

L. Qiu, W. Wang, and H. Zhang. *Smith normal form and the generalized spectral characterization of graphs.* Discrete Mathematics, 2023

Category-theoretic moment method:

W. Sawin and M.M. Wood. *The moment problem for random objects in a category.* arXiv:2210.06279v1, 2022.

Cokernel statistics of random matrices:

H.H. Nguyen and M.M. Wood. *Random integral matrices: universality of surjectivity and the cokernel.* Inventiones Mathematicae, 2022

G. Cheong and M. Yu. *The distribution of the cokernel of a polynomial evaluated at a random integral matrix.* arXiv:2303.09125v3, 2023

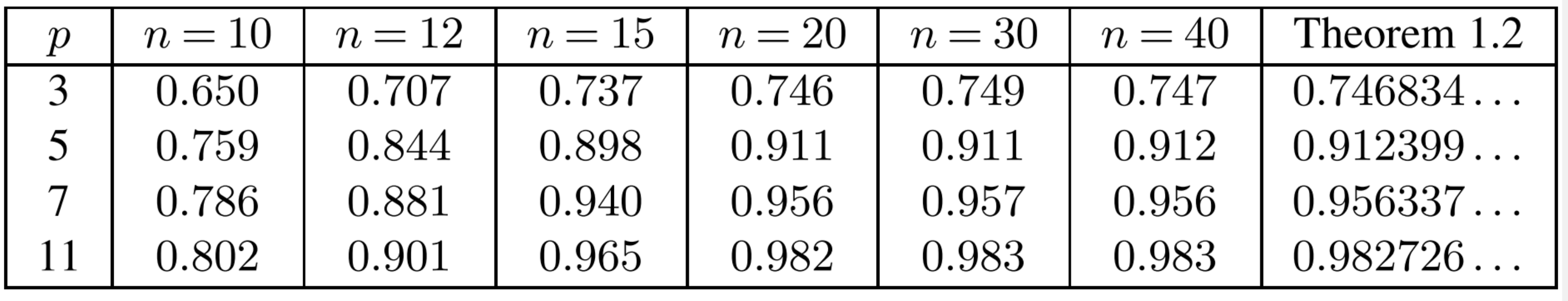
The current work:

A. Van Werde. *Cokernel statistics for walk matrices of directed and weighted random graphs.* arXiv:2401.12655, 2024

Feel free to contact me! **a.van.werde@tue.nl**

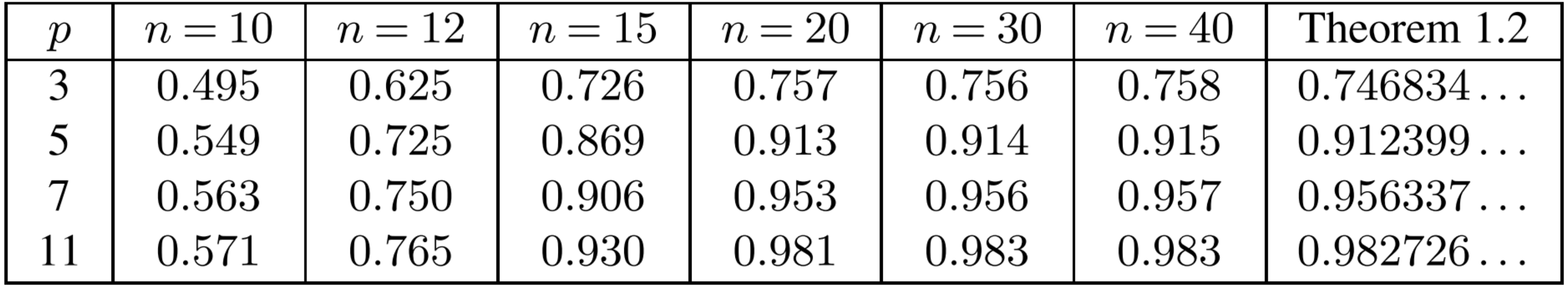
Numerical evidence

Estimated probability that $\operatorname{coker}(W)_{p^2} \in \{0, \mathbb{Z}/p\mathbb{Z}\}$ for $\boldsymbol{X} \sim \mathrm{Unif}\{0,1\}^{n\times n}$:



p	$n=10$	$n=12$	$n=15$	$n=20$	$n=30$	$n=40$	Theorem 1.2
3	0.650	0.707	0.737	0.746	0.749	0.747	0.746834...
5	0.759	0.844	0.898	0.911	0.911	0.912	0.912399...
7	0.786	0.881	0.940	0.956	0.957	0.956	0.956337...
11	0.802	0.901	0.965	0.982	0.983	0.983	0.982726...

Estimated probability that $\operatorname{coker}(W)_{p^2} \in \{0, \mathbb{Z}/p\mathbb{Z}\}$ for $\boldsymbol{X} = \boldsymbol{A}_G$ with $G \sim \mathcal{G}(n, 1/2)$:



p	$n=10$	$n=12$	$n=15$	$n=20$	$n=30$	$n=40$	Theorem 1.2
3	0.495	0.625	0.726	0.757	0.756	0.758	0.746834...
5	0.549	0.725	0.869	0.913	0.914	0.915	0.912399...
7	0.563	0.750	0.906	0.953	0.956	0.957	0.956337...
11	0.571	0.765	0.930	0.981	0.983	0.983	0.982726...