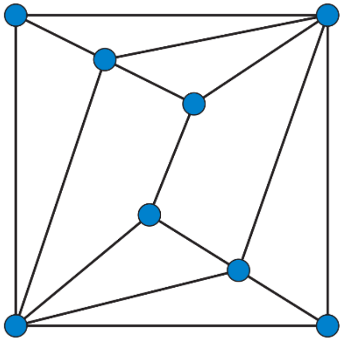
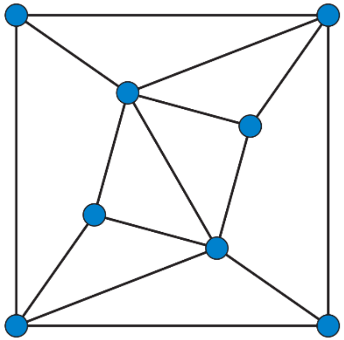
Cokernel statistics of walk matrices

Towards generalized spectral determinacy of random graphs





Alexander Van Werde, Dutch-Belgian discrete math seminar (2024)

Preprint available at arXiv:2401.12655

TU/e EINDHOVEN UNIVERSITY OF TECHNOLOGY

Can *all* information about a graph be recovered from its spectrum?

Some history: spectral determinacy of graphs

1957 (Collatz and Sinogowitz)

No, non-isomorphic graphs can have the same adjacency spectrum.

1973 (Schwenk)

Almost all trees are cospectral!

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"It is an open question whether almost all graphs are characterized by their characteristic polynomials. It is not even clear if we should seek to prove this, or to disprove it."

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2023 (Koval and Kwan)

At least $\exp(cn)$ graphs are determined by spectrum.

Some history: generalized spectral determinacy

1980 (Johnson and Newman)

"It is our view, however, that to some extent these examples are algebraic accidents due to the interpretation of the formal symbols 0 and 1 as real numbers."

Definition. (Generalized cospectral)

Graphs G, H are said to be *generalized cospectral if*

$$\mathrm{spec}\left(\mathrm{A}_{\mathrm{G}}^{x,y}\right) = \mathrm{spec}\left(\mathrm{A}_{\mathrm{H}}^{x,y}\right) \qquad \forall x, y \in \mathbb{R}.$$

where $A_G^{x,y}$ is the variant on the adjacency matrix with $1 \to x$ and $0 \to y$.

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Conjecture (Wang): Satisfied with non-vanishing probability!

Sufficient condition for generalized spectral determinacy

Definition (Walk matrix)

Given an integer matrix $\boldsymbol{X} \in \mathbb{Z}^{n\times n}$, consider the matrix

$$\boldsymbol{W} := [e, \boldsymbol{X}e, , \boldsymbol{X}^2 e, \dots, \boldsymbol{X}^{n-1} e]$$

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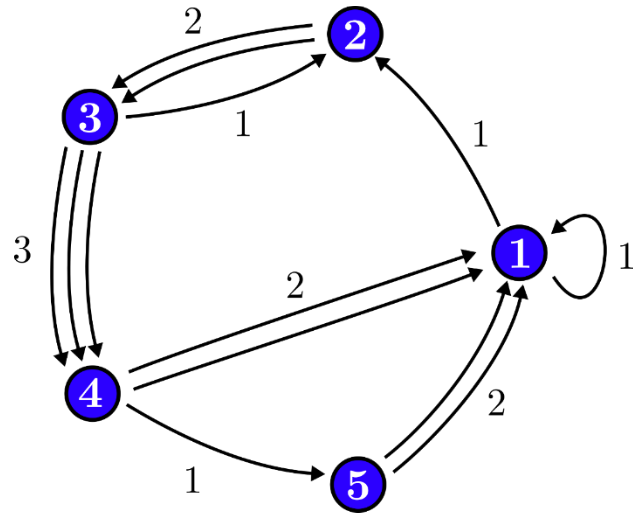
Interpret $\boldsymbol{X}_{i,j}$ as edge multiplicity.

Then, $\boldsymbol{W}_{i,j}$ counts walks of length $j - 1$ ending in i.

Example.

$$\boldsymbol{W}_{5,3} = 3 \cdot 1 = 3$$



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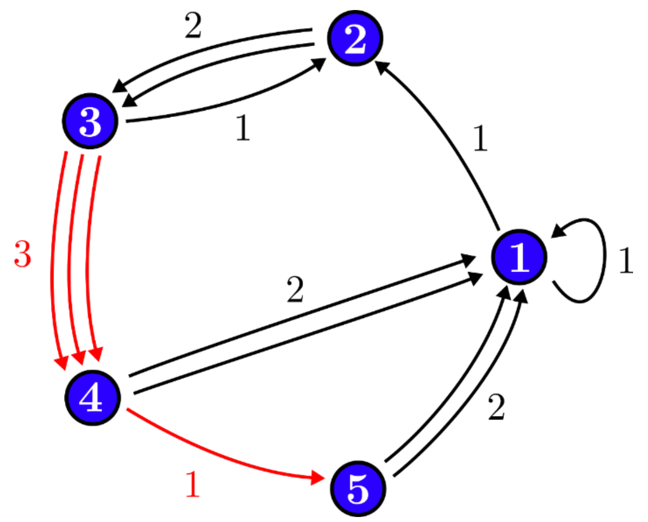

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Notation

$\mathrm{coker}(\boldsymbol{W}) := \mathbb{Z}^n / \boldsymbol{W}(\mathbb{Z}^n)$

Given an Abelian group G and a prime power p^m, let $G_{p^m} := G/p^m G$.

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Theorem. (Wang 2017; see also Qui, Wang, and Zhang 2023)

Consider a simple graph G and set $\boldsymbol{X} := \boldsymbol{A}_G$. Assume that $\operatorname{coker}(\mathbf{W})_{2^2} \cong (\mathbb{Z}/2\mathbb{Z})^{\lfloor n/2 \rfloor}$ and $\operatorname{coker}(\mathbf{W})_{p^2} \in \{0, \mathbb{Z}/p\mathbb{Z}\}$ for odd primes p.

Then, G is determined by generalized spectrum.

Suppose $\boldsymbol{X}$ is random.

How can we study the distribution of $\mathbf{coker}(\boldsymbol{W})$?

Results

Disclaimer.

For technical reasons, all results assume that $\boldsymbol{X}$ has independent entries.

This implies that we can not (yet) deal with the adjacency matrices of *simple* random graphs: those have dependent entries due to the symmetry constraint $\boldsymbol{X} = \boldsymbol{X}^{\mathrm{T}}$.

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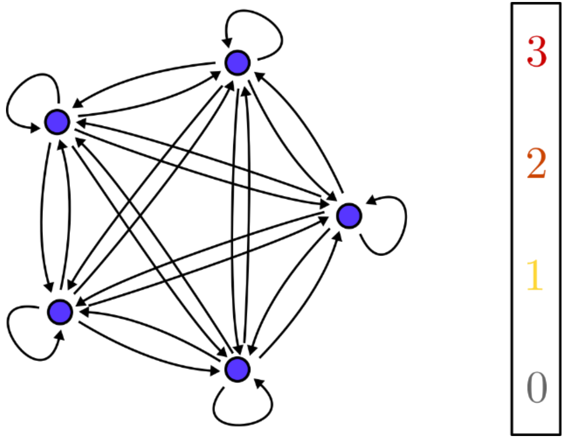
Assumption 1st result

Fix a prime p and integer $m \geq 0$.

Assume that the entries of $\boldsymbol{X}$ are independent and $\mathrm{Unif}\{0,1,\ldots,p^m - 1\}$-distributed.



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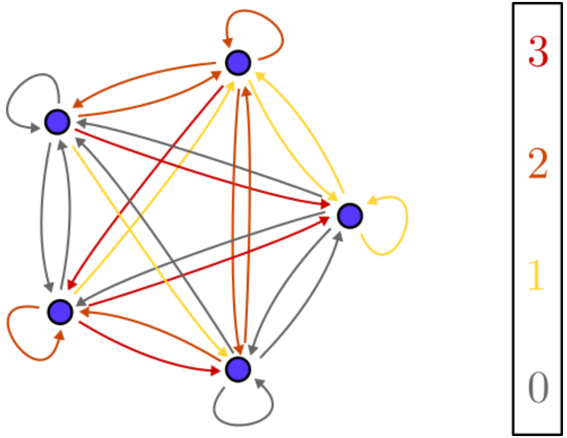

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Theorem 1.

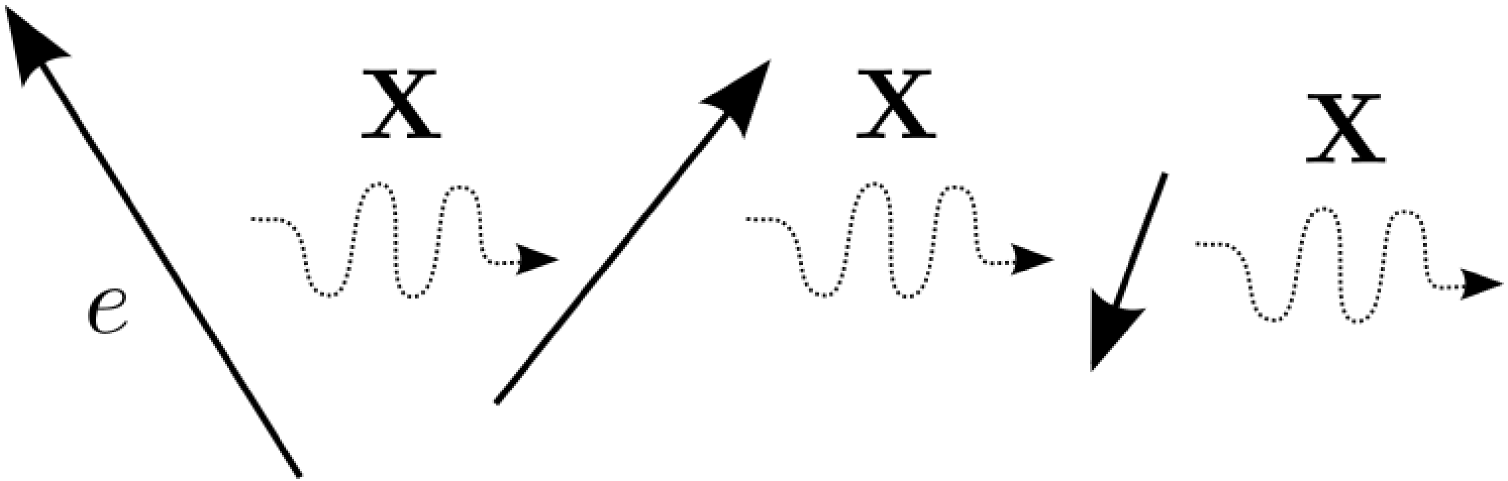
We have

$$\lim_{n\to\infty} \mathbb{P}\left(\mathrm{coker}(\boldsymbol{W})_{p^m} \cong \bigoplus_{i=1}^{\ell} \frac{\mathbb{Z}}{p^{\lambda_i}\mathbb{Z}}\right) = \Pi_{i=i_0}^{\infty}\left(1 - p^{-(i+1)}\right) \Pi_{j=1}^{\ell} p^{-j\delta_j}$$

for every $0 = \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_\ell \leq m$.

Here, $i_0 := \#\{i \leq \ell : \lambda_i = m\}$ and $\delta_j = \lambda_{\ell-j+1} - \lambda_{\ell-j}$.

Proof idea



Proof idea

There is dependence!

Observe that if

$$\boldsymbol{X}^j e \in \mathrm{span}_{\mathbb{Z}}(e, \boldsymbol{X}e, \dots, \boldsymbol{X}^{j-1}e) + p^k\mathbb{Z}^n$$

then also

$$\boldsymbol{X}^{j+1} e \in \mathrm{span}_{\mathbb{Z}}(e, \boldsymbol{X}e, \dots, \boldsymbol{X}^{j-1}e, \boldsymbol{X}^j e) + p^k\mathbb{Z}^n.$$

Key observation. (Informally)

Aside from the obstruction above, there is independence.

Interpretable proof!

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How can we study *unweighted* graphs?

Results

Assumption simplified 2nd result

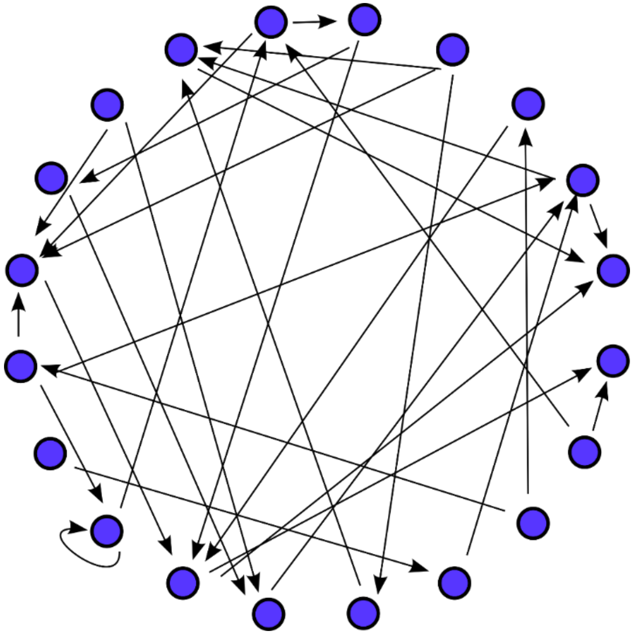
Suppose $\boldsymbol{X}$ has independent $\{0,1\}$-valued entries. (Not necessarily identically distributed.)

Further, consider a sparse setting:

$$\mathbb{P}(\boldsymbol{X}_{i,j} = 1) \leq \mathbb{P}(\boldsymbol{X}_{i,j} = 0)$$

But not *too* sparse:

$$\mathbb{P}(\boldsymbol{X}_{i,j} = 1) \gg \ln(n)/n$$



Results

Assumption simplified 2nd result

$\boldsymbol{X}$ has independent $\{0,1\}$-valued entries with $\mathbb{P}(\boldsymbol{X}_{i,j} = 1) \leq \mathbb{P}(\boldsymbol{X}_{i,j} = 0)$ and

$$\mathbb{P}\left(\boldsymbol{X}_{i,j} = 1\right) \gg \ln(n)/n$$

Technical condition.

Additionally assume tightness:

$$\lim_{C\to\infty} \liminf_{n\to\infty} \mathbb{P}\left(\#\mathrm{coker}(\boldsymbol{W})_{p^m} \leq C\right) = 1.$$

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Theorem 2. (Simplified)

Fix a finite collection of primes $\mathcal{P}$.

Then, given the conditions above,

1. The same limiting law applies to $\mathrm{coker}(\boldsymbol{W})_{p^m}$ for every $p \in \mathcal{P}$.
2. We have asymptotic independence for different primes $p \in \mathcal{P}$.

Robust proof technique:

category-theoretic moment method.

Category-theoretic moment method

Observation.

$\mathrm{coker}(\boldsymbol{W})$ is not only an Abelian group.

There is a canonical $\mathbb{Z}[x]$-module structure induced by the action of $\boldsymbol{X}$.

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Category-theoretic moment method (Sawin and Wood, 2022)

It suffices to determine

$$\lim_{n\to\infty} \mathbb{E}\left[\#\mathrm{Sur}_{\mathbb{Z}[x]}(\mathrm{coker}(\boldsymbol{W}), N)\right]$$

for every fixed finite $\mathbb{Z}[x]$-module N.

Future work

Extension to simple graphs.

The prime 2 behaves very different.

Odd primes are qualitatively similar. Numerics suggest a small quantitative difference.

Conjecture. (Technical condition is satisfied)

If $\boldsymbol{X}$ has independent $\{0,1\}$-valued entries with $\mathbb{P}(\boldsymbol{X}_{i,j} = 1) \leq \mathbb{P}(\boldsymbol{X}_{i,j} = 0)$ and $\mathbb{P}\left(\boldsymbol{X}_{i,j} = 1\right) \gg \ln(n)/n$.

Then,

$$\lim_{C\to\infty} \liminf_{n\to\infty} \mathbb{P}\left(\#\mathrm{coker}(\boldsymbol{W})_{p^m} \leq C\right) = 1.$$

Thank you!

Key reference related to this talk are as follows:

Generalized spectral determinacy:

W. Wang and C.-X. Xu. *A sufficient condition for a family of graphs being determined by their generalized spectra*. European Journal of Combinatorics, 2006.

W. Wang. *A simple arithmetic criterion for graphs being determined by their generalized spectra.* Journal of Combinatorial Theory, Series B, 2017.

L. Qiu, W. Wang, and H. Zhang. *Smith normal form and the generalized spectral characterization of graphs.* Discrete Mathematics, 2023

Category-theoretic moment method:

W. Sawin and M.M. Wood. *The moment problem for random objects in a category.* arXiv:2210.06279v1, 2022.

The current work:

A. Van Werde. *Cokernel statistics for walk matrices of directed and weighted random graphs.* arXiv:2401.12655, 2024

Feel free to contact me! **a.van.werde@tue.nl**

Numerical evidence

Estimated probability that $\mathrm{coker}(W)_{p^2} \in \{0, \mathbb{Z}/p\mathbb{Z}\}$ for $\boldsymbol{X} \sim \mathrm{Unif}\{0,1\}^{n\times n}$:

p	$n=10$	$n=12$	$n=15$	$n=20$	$n=30$	$n=40$	Theorem 1.2
3	0.650	0.707	0.737	0.746	0.749	0.747	0.746834...
5	0.759	0.844	0.898	0.911	0.911	0.912	0.912399...
7	0.786	0.881	0.940	0.956	0.957	0.956	0.956337...
11	0.802	0.901	0.965	0.982	0.983	0.983	0.982726...

Estimated probability that $\mathrm{coker}(W)_{p^2} \in \{0, \mathbb{Z}/p\mathbb{Z}\}$ for $\boldsymbol{X} = \boldsymbol{A}_G$ with $G \sim \mathcal{G}(n, 1/2)$:

p	$n=10$	$n=12$	$n=15$	$n=20$	$n=30$	$n=40$	Theorem 1.2
3	0.495	0.625	0.726	0.757	0.756	0.758	0.746834...
5	0.549	0.725	0.869	0.913	0.914	0.915	0.912399...
7	0.563	0.750	0.906	0.953	0.956	0.957	0.956337...
11	0.571	0.765	0.930	0.981	0.983	0.983	0.982726...