Towards generalized spectral determinacy of random graphs

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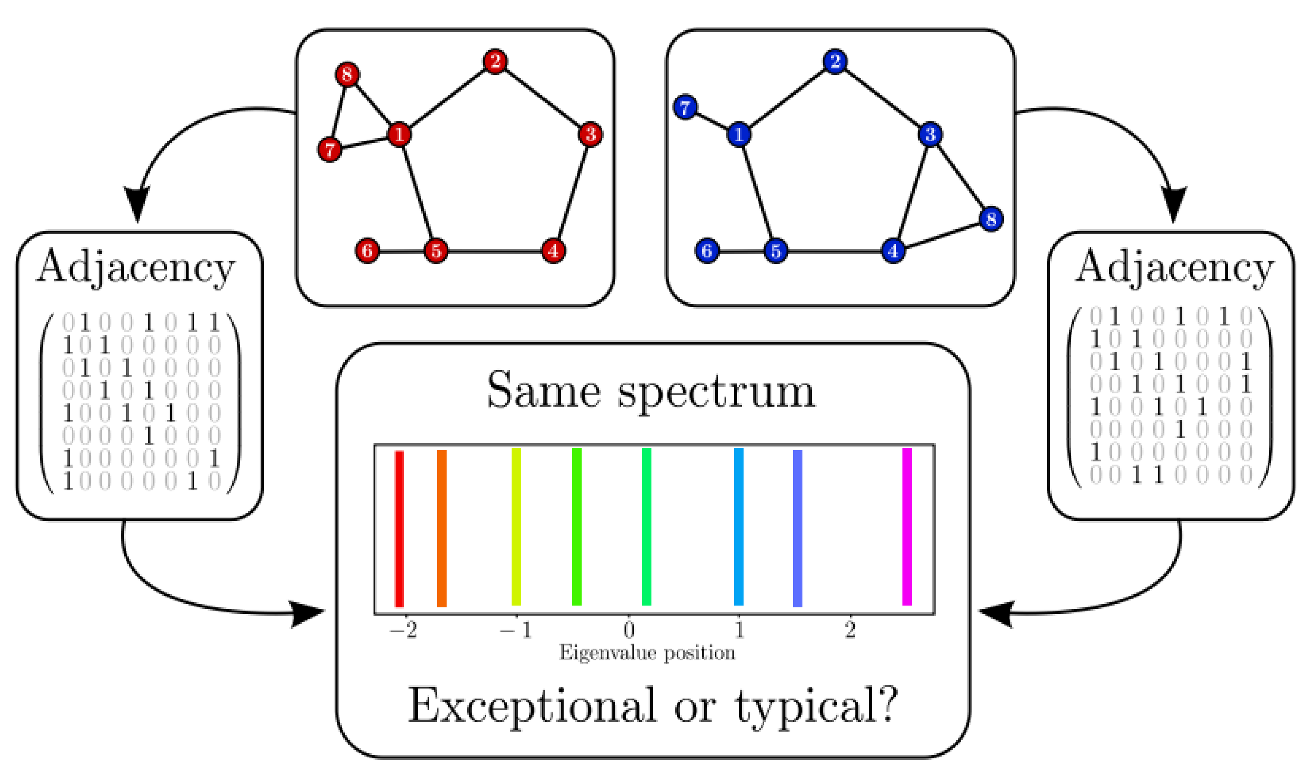
Spectral characterization

Surprisingly many graph-theoretic properties can be studied using eigenvalues of associated matrices. One may wonder how far one can take this perspective:

Is a graph typically characterized by spectral information?

Non-isomorphic graphs with identical adjacency spectra exist, but a fundamental conjecture by Van Dam and Haemers proposes that almost all graphs are characterized by spectrum [1].

Pathologically, though, there are many more known methods to prove that a graph is *not* characterized by spectrum than to show that it is. The conjecture remains wide-open…



Generalized spectrum

Johnson and Newman proposed in 1980 [2] that some of the difficulties surrounding spectral characterization may be symptoms of a deeper issue: taking $\{0,1\}$-valued entries in the adjacency matrix is arbitrary.

Given a graph G, they proposed to instead consider the family of matrices $\{A^{x,y}(G) : x, y \in \mathbb{R}\}$ with

$$A^{x,y}_{i,j}(G) := \begin{cases} x & \text{if } i,j \text{ are neighbors in } G, \\ y & \text{else.} \end{cases}$$

Graphs G and H are *generalized cospectral* if $A^{x,y}(G)$ and $A^{x,y}(H)$ have equal eigenvalues for every x, y.

Wang found sufficient conditions for generalized spectral characterization using the walk matrix [3]. However, prior work on the frequency of the condition's applicability was limited to numerical investigations, and it was not clear what proof techniques could be used.

Definition. Given $X \in \mathbb{Z}^{n\times n}$, the associated *walk matrix* W has columns $e, Xe, \dots, X^{n-1}e$ where $e = (1, \dots, 1)^T$.

Theorem. (Wang) Consider a simple graph, let X be the adjacency matrix, and consider the prime factorization

$$\det(W) = \pm p_1^{k_1} \cdots p_r^{k_r}.$$

If $k_i \le 1$ for all odd p_i and $k_i \le n/2$ for $p_i = 2$, then the graph is characterized by generalized spectrum.

A new abstract-algebraic perspective

If we choose the underlying graph uniformly at random, then by Wang's condition, we find the following:

To show that graphs are often characterized by generalized spectrum, it suffices to study the walk matrix generated by a random matrix X.

To make progress on this, it turns out to be more insightful to rephrase in abstract-algebraic language. Specifically, more refined information than the determinant is contained in the *cokernel* which is the Abelian group given by

$$\operatorname{coker}(W) := \mathbb{Z}^n / W(\mathbb{Z}^n) \quad \text{where} \quad W(\mathbb{Z}^n) := \{Wv : v \in \mathbb{Z}^n\}$$

For instance, $p^2 \nmid \det(W)$ if and only if $\operatorname{coker}(W)/p^2\operatorname{coker}(W) \in \{0, \mathbb{Z}/p\mathbb{Z}\}$. Thus, studying the satisfaction frequency of Wang's condition comes down to the following:

It suffices to study the law of the random group $\operatorname{coker}(W)$!

My results do exactly this in a simplified setting where X is taken to have independent entries. Thus, I studied random directed graphs instead of simple graphs.

The results are stated below, but the key contribution lies particularly in the development of new proof techniques. Specifically, I exploited the new abstract algebraic perspective with the following observation:

There is a recent proof technique, called the *category-theoretic moment method*, that can be used to study random abstract-algebraic objects!

Results

Theorem 1. (Cokernel statistics for uniform weights) Suppose that X has independent $\mathrm{Unif}\{0,1,\dots,p^m-1\}$-distributed entries. Then,

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\operatorname{coker}(W)}{p^m \operatorname{coker}(W)} \cong \oplus_{i=1}^{\ell} \frac{\mathbb{Z}}{p^{\lambda_i}\mathbb{Z}}\right) = \prod_{i=i_0}^{\ell}\left(1 - p^{-(i+1)}\right) \prod_{j=1}^{\ell} p^{-j\delta_j}$$

where $i_0 := \#\{i \le \ell : \lambda_i = m\}$ and $\delta_j = \lambda_{\ell-j+1} - \lambda_{\ell-j}$.

Theorem 2. (Unweighted graphs) Suppose that X has independent $\mathrm{Unif}\{0,1\}$-distributed entries. Then, given a mild tightness condition, for every fixed set of primes $\mathcal{P}$,

1. The same limiting law for p^m-quotients holds for $p \in \mathcal{P}$.
2. Asymptotic independence holds for different $p \in \mathcal{P}$.

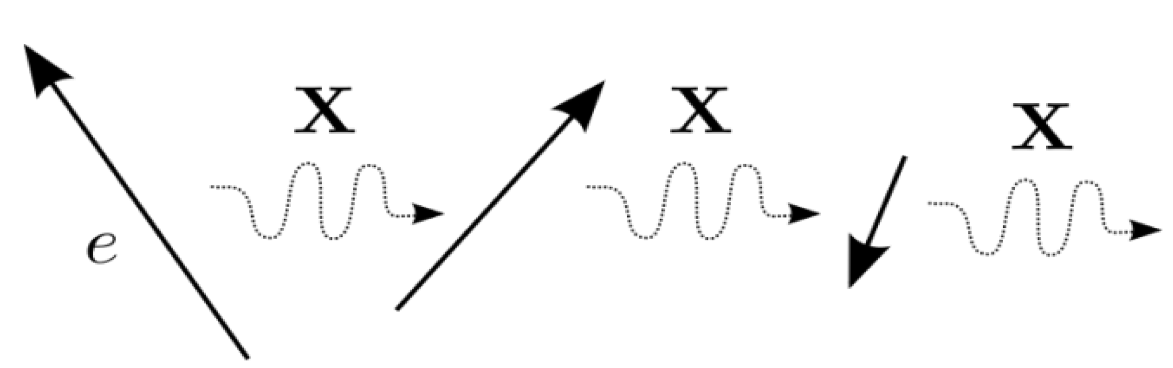
Proof techniques

Proof idea for Theorem 1: The assumption of uniformly distributed entries enables a direct analysis of the stochastic process $e, Xe, X^2e, \dots, X^{n-1}e$ as the next vector results from the uniform random action of X on the previous one.

These vectors are not independent since

$$X^j e \in \operatorname{span}_{\mathbb{Z}}(e, Xe, \dots X^{j-1}) + p^k\mathbb{Z}^n$$
$$\implies X^{j+1}e \in \operatorname{span}_{\mathbb{Z}}(e, Xe, \dots X^{j}) + p^k\mathbb{Z}^n$$

but this is the only obstruction to independence. Incorporating this obstruction with direct counting arguments yields the formula. □



Proof idea for Theorem 2: This is where we use the aforementioned *category-theoretic moment method [4].* In classical probability theory the moment method states that one can recover the distribution of a real-valued random variable Y based on its moments $\mathbb{E}[Y^n]$ given mild conditions. This helps because moments are often easy to estimate.

The category-theoretic moment method similarly states that the distribution of a random algebraic object Y can be characterized given the moments $\mathbb{E}[\#\mathrm{Sur}(Y,N)]$ where N runs over deterministic objects, say finite Abelian groups.

It turns out to be possible to compute the limiting category theoretic moments, although this is delicate. A key insight is that we need to pass to the category of $\mathbb{Z}[x]$-modules instead of only groups to simplify the computations. □

References. This poster is based on my paper "Cokernel statistics for walk matrices of directed and weighted random graphs" in *Combinatorics, Probability & Computing (2025)*. Also cited:

[1] Van Dam, E.R., and Haemers, W.H. "Which graphs are determined by their spectrum?" Linear Algebra Appl. (2003).

[2] Johnson C.R., Newman M. "A note on cospectral graphs", J. Combin. Theory B (1980)

[3] Wang, W. "Generalized spectral characterization revisited", Electrong. J. Combin. (2013)

[4] Sawin, W., and Wood M.M. "The moment problem for random objects in a category", arXiv preprint (2022)