Supplementary material for “Exact cospectrality probabilities for uniform random matrices”

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Abstract. A remark in the main paper states variants on the main results for matrices without symmetry constraint or for matrices with a Hermitian constraint. This supplemental gives the proofs, which are minor variations on those in the main document.

1. MATRICES WITHOUT SYMMETRY CONSTRAINT

Let R be a Dedekind domain and fix an ideal $I \subseteq R$ with R/I finite. Further, consider matrices $G_1, G_2 \in R^{n\times n}$.

Recall that every nontrivial quotient of a Dedekind domain is a principal ideal ring [3, p.278] and that this implies the existence of a unique Smith normal form [1, Theorem 15.24]. For every $j \in \{1,2\}$ there hence exist $U_j, V_j \in \mathrm{GL}_n(R/I)$ and $d_1^{(j)}, \ldots, d_n^{(j)} \in R$ with $d_{i+1}^{(j)}R + I \subseteq d_i^{(j)}R + I$ for every i such that

$$G_j \equiv U_j D_j V_j \bmod I \quad \text{with} \quad D_j := \mathrm{diag}(d_1^{(j)}, \ldots, d_n^{(j)}). \tag{1.1}$$

Recall that $\mathcal{N}_R(I) = \#R/I$ denotes the cardinality of the quotient ring.

Lemma 1.1. *Consider a random $n \times n$ matrix $Y \in R^{n\times n}$ whose entries $\{Y_{i,j} : i,j = 1,\ldots,n\}$ have independent and uniformly distributed reductions in R/I. Then, with $d_1^{(1)},\ldots,d_n^{(1)}$ and $d_1^{(2)},\ldots,d_n^{(2)}$ as in* (1.1),

$$\mathbb{P}\big(G_1 Y G_2 \equiv 0 \bmod I\big) = \prod_{i=1}^{n}\prod_{j=1}^{n} \frac{\#\{k \in R/I : k d_i^{(1)} d_j^{(2)} \equiv 0 \bmod I\}}{\mathcal{N}_R(I)}. \tag{1.2}$$

Proof. Note that the law of $Y \bmod I$ is characterized the fact that $Y + M \bmod I$ has the same law as Y for every deterministic M. Further, for every deterministic symmetric $M \in (R/I)^{n\times n}$,

$$V_1 Y U_2 + M \equiv V_1(Y + \tilde{S})U_2 \bmod I \quad \text{with} \quad \tilde{S} := V_1^{-1} S U_2^{-1}. \tag{1.3}$$

Consequently, the translation invariance of the law of $Y \bmod I$ implies that the law of $V_1 Y U_2 \bmod I$ is also translation invariant. Hence, $V_1 Y U_2 \bmod I$ has the same law as $Y \bmod I$. Now, using (1.1),

$$\mathbb{P}\big(G_1 Y G_2 \equiv 0 \bmod I\big) = \mathbb{P}(U_1 D_1 Y D_2 V_2 \equiv 0 \bmod I). \tag{1.4}$$

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The invertibility of $U_1, V_2 \in \mathrm{GL}_n(R/I)$ further implies that $U_1 D_1 Y D_2 V_2 \equiv 0 \bmod I$ if and only if $D_1 Y D_2$. Using the independence and uniformity of the entries now yields (1.2). □

Theorem 1.2. *With the assumptions and notation of Lemma 1.1,*

$$\mathbb{P}\big(G_1 Y G_2 \equiv 0 \bmod I\big) = \textstyle\prod_{i=1}^n \prod_{j=1}^n \mathcal{N}_R(d_i^{(1)} d_j^{(2)} R + I)/\mathcal{N}_R(I). \tag{1.5}$$

Proof. Recall from the main document [2] that for every $d \in R$,

$$\#\{k \in R/I : kd \equiv 0 \bmod I\} = \mathcal{N}_R(dR + I). \tag{1.6}$$

The result is hence immediate from Lemma 1.1. □

2. MATRICES WITH HERMITIAN CONSTRAINT

As before, we let R be a Dedekind domain and consider an ideal $I \subseteq R$ with R/I finite. Further, consider a ring automorphism $\tau : R \to R$ with $\tau \circ \tau = \mathrm{Id}_R$. Let $R_\tau := \{r \in R : \tau(r) = r\}$ be the subring of elements that are fixed by τ, and suppose that the ideal $I \subseteq R$ satisfies $\{\tau(i) : i \in I\} = I$.

Fix a matrix $G \in R^{n\times n}$. We denote the matrix found by the transpose and applying τ entry-wise by $G^\tau := (\tau(G_{j,i}))_{i,j=1}^n$.

Definition 2.1. A matrix $H \in R^{n\times n}$ is said to be *τ-Hermitian* if $H^\tau = H$.

Note that the diagonal entries of a τ-Hermitian matrix have values in R_τ.

Definition 2.2. A random τ-Hermitian matrix H is said to have the *uniform distribution modulo I* if it satisfies the following properties:

1. The upper-triangular entries $\{H_{i,j} : i \leq j\}$ are independent.
2. It holds for every $i < j$ that $H_{i,j} \bmod I$ is uniformly distributed in R/I.
3. It holds for every i that $H_{i,i} \bmod I \cap R_\tau$ is uniformly distributed on $R_\tau/(I \cap R_\tau)$.

As before, consider the Smith normal form of G over R/I. That is, there exist $U, V \in \mathrm{GL}_n(R/I)$ and $d_1, \ldots, d_n \in R$ with $d_{i+1}R + I \subseteq d_i R + I$ for every i such that

$$G \equiv UDV \bmod I \quad \text{with} \quad D := \mathrm{diag}(d_1, \ldots, d_n). \tag{2.1}$$

Lemma 2.3. *Consider a random τ-Hermitian random matrix $H \in R^{n\times n}$ with uniform distribution modulo I. Then, with $d_1, \ldots, d_n$ as in* (2.1),

$$\begin{aligned}\mathbb{P}\big(G^\tau H G \equiv 0 \bmod I\big) &= \prod_{i=1}^n \frac{\#\{k \in R_\tau/(I \cap R_\tau) : k\tau(d_i)d_i \equiv 0 \bmod I \cap R_\tau\}}{\mathcal{N}_{R_\tau}(I \cap R_\tau)} \\ &\quad \times \prod_{i=1}^{n-1} \prod_{j=i+1}^n \frac{\#\{k \in R/I : k\tau(d_i)d_j \equiv 0 \bmod I\}}{\mathcal{N}_R(I)}.\end{aligned} \tag{2.2}$$

Proof. It follows from the assumption that $I = \{\tau(i) : i \in I\}$ that τ descends to a ring automorphism on R/I. We again denote the latter automorphism as τ and let it be understood that the notation G^τ and definition of τ-Hermitianity may now also be applied to matrices over R/I.

Note that the law of $H \bmod I$ is characterized as the unique probability distribution on τ-Hermitian random matrices over R/I such that $H + M \bmod I$ has the same law as H for every deterministic τ-Hermitian $M \in R^{n\times n}$. Further, for every deterministic τ-Hermitian M,

$$U^\tau HU = U^\tau(H + \tilde{M})U \bmod I \quad \text{with} \quad \tilde{M} = (U^\tau)^{-1}MU^{-1}. \tag{2.3}$$

The assumption that $\tau \circ \tau = \mathrm{Id}_R$ here ensures that $U^\tau HU$ and $\tilde{M}$ are again τ-Hermitian matrices. In particular, the translation invariance of the law of $H \bmod I$ implies that $H + \tilde{M}$ is again a uniform τ-Hermitian random matrix. Using this in (2.3) proves $U^\tau HU$ is a translation invariant τ-Hermitian random matrix, implying that it has the same law as H.

Now, using (2.1),

$$\begin{aligned}\mathbb{P}\big(G^\tau HG \equiv 0 \bmod I\big) &= \mathbb{P}\big(V^\tau D^\tau U^\tau HUDV \equiv 0 \bmod I\big) \\ &= \mathbb{P}\big(V^\tau D^\tau HDV \equiv 0 \bmod I\big).\end{aligned} \tag{2.4}$$

Here, the invertibility of $V \in \mathrm{GL}_n(R/I)$ implies that $V^\tau D^\tau HDV \equiv 0 \bmod I$ if and only if $D^\tau HD \equiv 0 \bmod I$. The result now follows from the uniformity and independence of the entries. We here use that $\tau(d_i)d_i \in R_\tau$ by the assumption that $\tau \circ \tau = \mathrm{Id}_R$ when considering the diagonal entries, and we use that $\tau(d_i)d_j \equiv 0 \bmod I$ if and only if $\tau(d_j)d_i \equiv 0 \bmod I$ by the assumption that $I = \{\tau(i) : i \in I\}$ when considering the off-diagonal entries. □

Theorem 2.4. *With the assumptions and notation of Lemma 2.3,*

$$\mathbb{P}\big(G^\tau HG \equiv 0 \bmod I\big) = \Bigg(\prod_{i=1}^{n} \frac{\mathcal{N}_{R_\tau}(\tau(d_i)d_iR_\tau + I \cap R_\tau)}{\mathcal{N}_{R_\tau}(I \cap R_\tau)}\Bigg)\Bigg(\prod_{i=1}^{n-1}\prod_{j=i+1}^{n} \frac{\mathcal{N}_R(\tau(d_i)d_jR + I)}{\mathcal{N}_R(I)}\Bigg). \tag{2.5}$$

Proof. This is immediate from Lemma 2.3 by using that for every $d \in R$,

$$\#\{k \in R/I : kd \equiv 0 \bmod I\} = \mathcal{N}_R(dR + I), \tag{2.6}$$

and that for every $d_\tau \in R_\tau$,

$$\#\{k \in R_\tau/(I \cap R_\tau) : k\tau(d_i)d_i \equiv 0 \bmod I \cap R_\tau\} = \mathcal{N}_{R_\tau}(d_\tau R_\tau + I \cap R_\tau). \tag{2.7}$$

□

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